Subbotin Stein envelopes and factors: ordinary and weighted bounds by test class

noninteger powers above one

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Abstract

We organise pointwise envelopes and uniform factors for the Subbotin Stein equation by test class: Kolmogorov, bounded measurable, Wasserstein, Fortet–Mourier, and integer-order Zolotarev. The first section gives the bounds at derivative orders zero, one, two, and general order, with separate ordinary and weighted factors. Integrated tails and weighted Mills ratios give exact envelopes at the available single-seminorm orders. Polynomial and localised smooth examples explain unavailable bounds. The parameter regime and the behaviour at the origin are specified explicitly. The constant operator weight is kept separate from the covariance Stein kernel, and the latter detects the polynomial obstruction to higher single-seminorm bounds. Exact supremum formulas are distinguished from explicit upper bounds and numerical illustrations. The structure and conventions follow the Gaussian, Gamma and Beta companion papers.

Keywords: Stein equation; Stein factors; weighted derivative bounds; kernel representation; probability metrics.

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1 Results at a glance and by test class

1.1 Equation, normalisation, and the meaning of a weighted factor

Let $\beta > 1$, $\beta \notin \mathbb{N}$ and $\lambda > 0$. Following the flexible parametrisation in Eichelsbacher [2], set

$$p(x) = \frac{\beta\lambda^{1/\beta}}{2\Gamma(1/\beta)} e^{-\lambda|x|^\beta}, \qquad I = (\ell, u) = \mathbb{R}. \tag{1.1}$$

His analysis uses $\beta > 1$; the scaled density itself is well-defined for every $\beta > 0$. The Rapin–Swan normalisation for $\beta > 1$ is $\lambda = 1/[\beta(\beta-1)]$. That substitution is invalid for $\beta \leq 1$; there we keep $\lambda > 0$ explicitly. For $x \neq 0$, put $g(x) = \lambda\beta \operatorname{sgn}(x)|x|^{\beta-1}$. We use the density equation

$$f_h'(x) - g(x) f_h(x) = h(x) - \mathbb{E}h(Z), \qquad w = 1. \tag{1.2}$$

Thus $a = 1$, $d = g$. The covariance Stein kernel, which is an envelope here rather than the operator weight, is

$$\tau(x) = \frac{\lambda^{-2/\beta}}{\beta} e^{\lambda|x|^\beta} \Gamma(2/\beta, \lambda|x|^\beta), \qquad \tau(0) = \frac{\lambda^{-2/\beta}\Gamma(2/\beta)}{\beta}. \tag{1.3}$$

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The two half-lines are smooth. At zero the potential is $C^{\lfloor\beta\rfloor}$, not $C^{\lceil\beta\rceil}$. The canonical solution for a smooth test generically contains a nonzero multiple of $|x|^\beta$. All-order Kolmogorov assertions are therefore branch assertions at zero. The distinction between $1 < \beta < 2$ and $\beta > 2$ is explicit below.

For $\mathbb{E}|h(Z)| < \infty$, the canonical solution is

$$f_h(x) = \frac{\int_\ell^x \{h(t) - \mathbb{E}h(Z)\}p(t)\,dt}{w(x)p(x)} = -\frac{\int_x^u \{h(t) - \mathbb{E}h(Z)\}p(t)\,dt}{w(x)p(x)}. \tag{1.4}$$

The endpoint flux wpf_h vanishes at both ends. This specifies the solution; adding a constant to h leaves f_h unchanged. Write $F(x) = \Pr(Z \le x)$ and $Q = 1 - F$. All norms are over the support, with the one-sided conventions specified below at exceptional points. For bounded tests put $H_0 = \operatorname{osc}(h) = \sup h - \inf h$. For $r \ge 1$, $H_r = \|h^{(r)}\|_\infty < \infty$ means $h \in C^{r-1}$, with locally absolutely continuous $(r-1)$st derivative and essentially bounded rth derivative. Integrability is a separate requirement.

An *envelope* bounds $|f_h^{(n)}(x)|$ over the indicated normalised class. An exact envelope is the pointwise class supremum, with limiting extremisers allowed. Its ordinary factor is its spatial supremum; a weighted factor is the supremum after multiplication by a specified nonnegative weight ω. Thus $\omega f_h^{(n)}$ is a derivative multiplied by a weight, not $(\omega f_h)^{(n)}$. Throughout the tables, use the explicit evaluation rules

$$\mathfrak{C}[E] := \sup_{x\in I} E(x), \qquad \mathfrak{C}_\omega[E] := \sup_{x\in I} \omega(x)E(x). \tag{1.5}$$

These are one-dimensional supremum formulas for the factors, not unspecified constants. An entry ∞ means that a pointwise envelope exists but has infinite spatial supremum. "None" denotes failure of a finite class envelope on an open set of interior points. "Not defined" means that the full test class contains nonintegrable functions.

The quantities needed in the tables. Set $P_0 = Q_0 = p$ and, whenever finite,

$$P_j(x) = \int_\ell^x P_{j-1}(t)\,dt, \qquad Q_j(x) = \int_x^u Q_{j-1}(t)\,dt \quad (j \ge 1). \tag{1.6}$$

Thus $P_1 = F$, $Q_1 = Q$. Define

$$a(x) = 1/w(x), \qquad d(x) = -\frac{(wp)'(x)}{w(x)p(x)},$$
$$S(x) = \frac{F(x)}{w(x)p(x)}, \qquad R(x) = \frac{Q(x)}{w(x)p(x)}, \tag{1.7}$$
$$A(x) = \frac{F(x)Q(x)}{w(x)p(x)}, \qquad K_n(x) = \max\{Q(x)|S^{(n)}(x)|, F(x)|R^{(n)}(x)|\}, \tag{1.8}$$

$$B_1(x) = \frac{a(x) + F(x)|R'(x)| + Q(x)|S'(x)|}{2}. \tag{1.9}$$

In particular $K_0 = A$. When $\mathbb{E}|Z| < \infty$, also set

$$T(x) = R(x)P_2(x) + S(x)Q_2(x) = \tau(x)/w(x), \tag{1.10}$$
$$W_1(x) = |R'(x)|P_2(x) + |S'(x)|Q_2(x), \tag{1.11}$$
$$W_2(x) = a(x) + |R''(x)|P_2(x) + |S''(x)|Q_2(x). \tag{1.12}$$

Here $\tau p = \int_\ell^x (\mu - t)p(t)\,dt$ and $\mu = \mathbb{E}Z$. No higher-order moment is needed for W_2. For direct calculation, only two differentiations are needed:

$$\begin{aligned} S' &= a + dS, & R' &= -a + dR, \\ S'' &= a' + da + (d' + d^2)S, & R'' &= -a' - da + (d' + d^2)R. \end{aligned} \tag{1.13}$$

For $r \geq 1$ with finite rth absolute moment, write

$$E_{n,r}(x) = |R^{(n)}(x)|P_{r+1}(x) + |S^{(n)}(x)|Q_{r+1}(x) + \mathbf{1}_{\{n=r+1\}}a(x), \quad 0 \leq n \leq r+1. \tag{1.14}$$

This last envelope applies only when the polynomial cancellation condition in the Zolotarev subsection holds. In particular it must not automatically be called a diagonal Zolotarev envelope.

Ordinary factors and the main weight. The ordinary Kolmogorov factors are $u_0 = \mathfrak{C}[A] < \infty$ at order zero, 1 at order one, and ∞ at every order $n \geq 2$. The bounded-test factors are u_0 and 1 at orders zero and one. For $1 < \beta < 2$ the ordinary Wasserstein factors are ∞, c_1, ∞, where $c_1 = \mathfrak{C}[W_1] < \infty$. For $\beta > 2$ they are $\tau(0), c_1, c_2$, where $c_2 = \mathfrak{C}[W_2] < \infty$. The common weight ω_* below gives finite factors for all three low-order envelopes in both regimes.

For a single weight covering the first three orders, put

$$\begin{aligned} \rho = \lambda^{1/\beta}|x|, \quad v = (2-\beta)_+, \quad M = \max\{2-\beta, \beta-1, 0\}, \\ \omega_*(x) = \left(\frac{\rho}{1+\rho}\right)^v (1+\rho)^{-M}. \end{aligned} \tag{1.15}$$

Take $\omega = \omega_*$ in the weighted columns. Its exact factors are $\mathfrak{C}_\omega[A]$, $\mathfrak{C}_\omega[B_1]$, $\mathfrak{C}_\omega[K_1]$, $\mathfrak{C}_\omega[K_2]$, $\mathfrak{C}_\omega[T]$, $\mathfrak{C}_\omega[W_1]$ and $\mathfrak{C}_\omega[W_2]$; all seven are finite. This is a useful explicit common weight, not a claim that its exponents are optimal for each individual entry. When $v = 0$ the first factor is identically one, including at zero. For higher Kolmogorov orders, a finite factor is obtained with

$$\omega_n(x) = \left(\frac{\rho}{1+\rho}\right)^{(n-\beta)_+} (1+\rho)^{-(n-1)(\beta-1)}, \qquad n \geq 1. \tag{1.16}$$

For integer powers the first factor can be omitted: the one-sided Mills derivatives are bounded at zero. These weights multiply branch derivatives and do not remove distributional atoms.

The exact order-zero factor. For $1 < \beta < 2$ put $a_* = 1/\beta$ and $P_{a_*}(u) = \gamma(a_*, u)/\Gamma(a_*)$. Eichelsbacher's Proposition 8.5 [2] identifies the unique positive root u_β of

$$2P_{a_*}(u)P'_{a_*}(u) = 1 - P_{a_*}(u)^2.$$

The exact factor is

$$u_0 = \frac{\Gamma(1/\beta)}{2\beta\lambda^{1/\beta}} e^{u_\beta}\{1 - P_{a_*}(u_\beta)^2\}.$$

There are two maximisers, at $x = \pm(u_\beta/\lambda)^{1/\beta}$. For $\beta > 2$ the maximum is at zero and $u_0 = 1/[4p(0)] = \Gamma(1/\beta)/(2\beta\lambda^{1/\beta})$ by the same paper's Section 8. This optimisation input is separate from the kernel derivation of the envelope A.

1.2 Kolmogorov tests: piecewise derivatives and the required weights

Let $h_z(t) = \mathbf{1}_{\{t \le z\}}$ and $f_z = f_{h_z}$.

Theorem 1.1 (Kolmogorov envelopes and factors)**.** *For every order for which the branch derivative is defined,*

$$f_z^{(n)}(x) = \begin{cases} Q(z)S^{(n)}(x), & x < z, \\ F(z)R^{(n)}(x), & x > z. \end{cases}$$

The exact pointwise envelopes and factors are

Order	Exact envelope	Ordinary factor	ω-weighted factor
0	A	$\mathfrak{C}[A]$	$\mathfrak{C}_\omega[A]$
1	K_1	$\mathfrak{C}[K_1]$	$\mathfrak{C}_\omega[K_1]$
2	K_2	$\mathfrak{C}[K_2]$	$\mathfrak{C}_\omega[K_2]$
$n \ge 0$	K_n	$\mathfrak{C}[K_n]$	$\mathfrak{C}_\omega[K_n]$

The solution is continuous at the threshold and $f_z'(z+) - f_z'(z-) = -a(z)$ at regular points. Derivatives in this table are branch derivatives; distributional derivatives can contain atoms.

1.3 Bounded measurable tests

The class $\operatorname{osc}(h) \le 1$ induces total variation in the convention $d_{\mathrm{TV}}(X, Z) = \sup_B |\Pr(X \in B) - \Pr(Z \in B)|$. The class $\|h\|_\infty \le 1$ induces twice that distance.

Theorem 1.2 (Bounded-test envelopes and factors)**.** *Let $H_0 = \operatorname{osc}(h) < \infty$. The following finite bounds are sharp:*

Order	Pointwise bound	Ordinary factor	ω-weighted factor
0	$A(x)H_0$	$\mathfrak{C}[A]$	$\mathfrak{C}_\omega[A]$
1	$B_1(x)H_0$ a.e.	$\mathfrak{C}[B_1]$	$\mathfrak{C}_\omega[B_1]$
2	no finite class envelope	none	none
$n \ge 2$	no finite class envelope	none	none

Factors in the last two columns multiply H_0. If $\|h\|_\infty \le B$, replace H_0 by $2B$; the resulting single-norm bounds are also sharp.

The higher-order negative statements hold even for smooth bounded tests. A weight which is positive near a regular interior point cannot restore these missing bounds. This differs from Kolmogorov tests, whose special piecewise form permits derivatives of all orders on smooth branches.

1.4 Wasserstein tests

Let $L = \mathrm{Lip}(h) = \|h'\|_\infty$. Assume the target has a finite first moment. The unit class induces Wasserstein distance when the compared variable also has a finite first moment.

Theorem 1.3 (Wasserstein envelopes and factors)**.** *On each regular open interval the solution belongs to C^1, its first derivative is locally absolutely continuous, and*

Order	Exact envelope times L	Ordinary factor	ω-weighted factor
0	$T(x)L$	$\mathfrak{C}[T]$	$\mathfrak{C}_\omega[T]$
1	$W_1(x)L$	$\mathfrak{C}[W_1]$	$\mathfrak{C}_\omega[W_1]$
2	$W_2(x)L$ a.e.	$\mathfrak{C}[W_2]$	$\mathfrak{C}_\omega[W_2]$
$n \geq 3$	no finite class envelope	none	none

The finite envelopes and factors are exact; factors multiply L.

The order-zero extremiser is $h(t) = t$, giving $f_h = -T$. For order one choose h' to match the signs of the two kernel pieces. For order two its value at the evaluation point can additionally be chosen on a shrinking interval, giving the extra term a in W_2.

1.5 Fortet–Mourier tests: size and Lipschitz constraints together

First use the rectangular convention $\mathcal{H}_{B,L} = \{h : \|h\|_\infty \leq B,\ \mathrm{Lip}(h) \leq L\}$. The usual rectangular unit class has $B = L = 1$. When the target has a first moment put

$$V_0 = \min\{2BA, LT\}, \quad V_1 = \min\{2BB_1, LW_1\}, \quad V_2 = LW_2.$$

The following table gives upper bounds; the first two minima are not asserted to be exact under both constraints:

Order	Pointwise upper bound	Ordinary upper factor	Weighted upper factor
0	V_0	$\mathfrak{C}[V_0]$	$\mathfrak{C}_\omega[V_0]$
1	V_1	$\mathfrak{C}[V_1]$	$\mathfrak{C}_\omega[V_1]$
2	V_2 a.e.	$\mathfrak{C}[V_2]$	$\mathfrak{C}_\omega[V_2]$
$n \geq 3$	no finite class envelope	none	none

Without a first moment, use $V_0 = 2BA$, $V_1 = 2BB_1$ and the valid replacement

$$V_2 = aL + 2B|a' + da| + 2BA|d' + d^2|. \tag{1.17}$$

This replacement is also available with a first moment: take the minimum of the two order-two upper bounds.

Exact refinements at orders zero and one. At order zero an exact one-parameter formula is

$$\max_{-B \leq c \leq B} \left\{ R(x) \int_\ell^x p(t)\, \mathrm{clip}_{[-B,B]}(c + L(x-t))\, dt - S(x) \int_x^u p(t)\, \mathrm{clip}_{[-B,B]}(c + L(x-t))\, dt \right\}. \tag{1.18}$$

When $R' \leq 0 \leq S'$, the exact first-order envelope is

$$-R'(x) \int_\ell^x p(t) \min\{2B, L|t-x|\}\, dt + S'(x) \int_x^u p(t) \min\{2B, L|t-x|\}\, dt. \tag{1.19}$$

Its factors are obtained by the same supremum rules (1.5). When these signs fail, the exact first-order envelope is the bounded Lipschitz norm of the explicitly known zero-mass measure

$$R'(x)p(t)\mathbf{1}_{t<x}\,dt - S'(x)p(t)\mathbf{1}_{t>x}\,dt + a(x)\delta_x(dt);$$

the preceding table remains an immediately usable upper bound. For the alternative sum norm $\|h\|_\infty + \mathrm{Lip}(h) \le 1$, take the supremum of the exact rectangular envelope over $B + L \le 1$. It is a different normalisation; rectangular factors must not be called sum-norm sharp.

1.6 Integer-order Zolotarev tests

Let $\mathcal{Z}_r = \{h : H_r \le 1\}$ without anchoring lower derivatives. All target moments exist. The exact low-order classification is

Class	Order 0	Order 1	Order 2
$\mathcal{Z}_1$	τ	W_1	W_2
$\mathcal{Z}_r$, $r \ge 2$	none	none	none

For general solution order, $\mathcal{Z}_1$ has no class envelope at $n \ge 3$. For every $r \ge 2$ there is no finite envelope on an open set at any solution order n. The exact factors of every finite entry E are $\mathfrak{C}[E]$ and $\mathfrak{C}_\omega[E]$; multiply by H_1 outside the unit class. The obstruction is already the linear test: $f_t = -\tau$, and τ is not a polynomial. Its derivatives are not identically zero on either open half-line. Arbitrary multiples of t have $H_r = 0$ for $r \ge 2$. An isolated zero of a derivative is not a uniform class bound; the pointwise criterion at such a point is Theorem 2.1.

The associated ideal metric is finite for laws with matching moments through degree $r-1$ and finite rth absolute moments. This metric condition does not eliminate polynomial freedom in individual Stein solutions. Fractional-order and anchored classes are different normalisations and are not being substituted for $\mathcal{Z}_r$ here.

1.7 Combining smoothness controls and choosing other weights

Each term below is available under its own regularity and integrability assumption; omit terms for which the indicated control is unavailable:

$$\begin{aligned} |f_h(x)| &\le \min\{A(x)H_0, T(x)H_1\}, \\ |f_h'(x)| &\le \min\{B_1(x)H_0, W_1(x)H_1\}. \end{aligned} \tag{1.20}$$

An available Zolotarev entry $E_{n,r}H_r$ can be added to the minimum for solution order n. Multiply an envelope by the desired weight before taking its supremum. Each single-seminorm envelope is sharp; sharpness of its minimum under simultaneous constraints is not asserted.

There is also an explicit all-order bound retaining lower controls. Define $A_1 = d$, $b_{1,0} = a$ and recursively, with missing coefficients zero,

$$\begin{aligned} A_{n+1} &= A_n' + dA_n, \\ b_{n+1,0} &= b_{n,0}' + aA_n, \qquad & b_{n+1,j} &= b_{n,j}' + b_{n,j-1} \quad (1 \le j \le n). \end{aligned} \tag{1.21}$$

Then, at every regular point,

$$f_h^{(n)} = A_n f_h + b_{n,0}(h - \mathbb{E}h(Z)) + \sum_{j=1}^{n-1} b_{n,j} h^{(j)}. \tag{1.22}$$

Consequently a valid envelope is

$$|A_n|AH_0 + |b_{n,0}|H_0 + \sum_{j=1}^{n-1} |b_{n,j}|H_j.$$

For Lipschitz, possibly unbounded tests replace its first two terms by $\{|A_n|T+|b_{n,0}|\mathbb{E}|Z-x|\}H_1$. These are upper bounds, not exact single-seminorm factors.

2 The kernel calculus

2.1 Integrated tails and weighted Mills ratios

The tails have the positive-integral representations

$$P_j(x) = \frac{\mathbb{E}(x-Z)_+^{j-1}}{(j-1)!}, \qquad Q_j(x) = \frac{\mathbb{E}(Z-x)_+^{j-1}}{(j-1)!} \quad (j \geq 1).$$

They are finite whenever the $(j-1)$st absolute moment exists. The derivatives of S, R can be computed without numerical differentiation of a quotient: start from (1.13) and differentiate recursively. No assumption of complete monotonicity is made when it has not been proved.

2.2 The array, its signs, and its three special representations

For $i, j \geq 0$ and regular x, define

$$K^{i,j}(x,t) = \frac{(-1)^j R^{(i)}(x) P_j(t) \mathbf{1}_{t \leq x} - S^{(i)}(x) Q_j(t) \mathbf{1}_{t>x}}{p(t)}. \tag{2.1}$$

For bounded tests,

$$\begin{aligned} f_h &= \mathbb{E}[K^{0,0}(x,Z)h(Z)], \\ f_h' &= \mathbb{E}[K^{1,0}(x,Z)h(Z)] + a(x)h(x). \end{aligned} \tag{2.2}$$

The first signed kernel has zero mean; the second has mean $-a$. For Lipschitz tests with a finite first target moment,

$$\begin{aligned} f_h &= -R\int_\ell^x F(t)h'(t)\,dt - S\int_x^u Q(t)h'(t)\,dt, \\ f_h' &= -R'\int_\ell^x F(t)h'(t)\,dt - S'\int_x^u Q(t)h'(t)\,dt, \\ f_h'' &= -R''\int_\ell^x F(t)h'(t)\,dt - S''\int_x^u Q(t)h'(t)\,dt + ah'(x). \end{aligned} \tag{2.3}$$

In particular $\mathbb{E}|K^{n,r}(x,Z)| = |R^{(n)}|P_{r+1} + |S^{(n)}|Q_{r+1}$. The last pointwise term in (2.3) must be retained.

2.3 Representations at every derivative order

Put $m_j(t) = t^j/j!$ and $\pi_{n,j}(x) = f_{m_j}^{(n)}(x)$.

Theorem 2.1 (The polynomial test for a single-seminorm bound)**.** *Suppose the rth absolute moment is finite and x is a regular point. For $0 \leq n \leq r+1$, a finite exact envelope depending only on H_r exists at x if and only if*

$$\pi_{n,j}(x) = 0 \quad (1 \leq j < r). \tag{2.4}$$

When this holds, the envelope for $H_r \leq 1$ is exactly $E_{n,r}(x)$. For $n \geq r+2$, no finite envelope exists at a regular point where $a > 0$.

Proof. Necessity follows by multiplying any offending polynomial by an arbitrary constant: its rth derivative vanishes. For sufficiency use the integral Taylor formula, subtracting its polynomial of degree $r-1$. The functional $h \mapsto f_h^{(n)}(x)$ annihilates that polynomial by (2.4). For a truncated power $(t-v)_+^{r-1}/(r-1)!$, its image is the kernel $(-1)^r R^{(n)}(x)P_r(v)$ when $v < x$ and $-S^{(n)}(x)Q_r(v)$ when $v > x$. At $n = r+1$ there is in addition $a(x)\delta_x(dv)$. The total variation is precisely (1.14). The integral Taylor identity can first be applied to compactly supported highest derivatives and then passed to the limit; the finite rth moment provides domination of the tails. Choosing $h^{(r)}$ to match the density signs, and choosing its value on a shrinking interval at x independently of the rest, proves sharpness. Smoothing yields the same suprema. The high-order failure follows from the local construction in Section 4. □

Thus finite integrated tails alone do not imply a valid Zolotarev bound. The polynomial test is what detects the correction terms.

3 Proofs of the envelopes, weights, and sharpness

3.1 Endpoint asymptotics and finiteness of the shape constants

For $x > 0$, repeated integration by parts in the stretched-exponential tail gives

$$\frac{Q(x)}{p(x)} \sim \frac{x^{1-\beta}}{\lambda\beta}, \qquad \tau(x) \sim \frac{x^{2-\beta}}{\lambda\beta}, \qquad Q_2(x) \sim \frac{p(x)}{g(x)^2}.$$

The symmetric statements hold in the other tail. Differentiation of the full tail expansion gives, for fixed $n \geq 1$, $K_n \asymp |x|^{(n-1)(\beta-1)}$. For $\beta > 1$, $W_1 \sim |x|^{1-\beta}/\lambda$; for $0 < \beta < 1$, $W_1 \sim (2-\beta)|x|^{1-\beta}/(\lambda\beta)$. For every $\beta > 0$, $W_2 \to 2$ in the tails. Near zero and for $\beta < 2$, $\beta \neq 1$,

$$W_2(x) \sim \lambda\beta|\beta-1|\tau(0)|x|^{\beta-2}.$$

For $0 < \beta < 1$ one also has $W_1 \sim \lambda\beta\tau(0)|x|^{\beta-1}$, $B_1 \asymp K_1 \asymp |x|^{\beta-1}$. For noninteger β and $n > \beta$, the leading singular term in $S^{(n)}, R^{(n)}$ has order $|x|^{\beta-n}$. For integer β, each half-line has an analytic extension to zero, so all one-sided derivative limits are finite. These facts prove the stated ordinary-factor classification and the sufficiency of the weights.

For $\beta \geq 1$, log-concavity gives $R' \leq 0 \leq S'$: on $x > 0$, $g(t) \geq g(x)$ yields $Q/p \leq 1/g(x)$, and reflection treats the other side. Thus $B_1 = 1$ and $K_1 \leq 1$; the tail limit of K_1 is 1, proving its sharp factor. Furthermore,

$$\tau(x) = \frac{1}{2}\int_0^\infty e^{-\lambda\{(x^2+v)^{\beta/2}-|x|^\beta\}}\,dv.$$

For $\beta \geq 2$ the integrand decreases as $|x|$ increases; hence $\sup\tau = \tau(0)$. For $\beta < 2$ the tail asymptotic makes this supremum infinite. For the convex-potential regimes, optimisation of A can also use [2]; that input is separate from these asymptotic calculations.

3.2 Polynomial obstruction and regularity at zero

The linear test has canonical solution $f_t = -\tau$. Except at $\beta = 1, 2$, this is not a polynomial on either half-line: for $\beta > 2$ it tends to zero at infinity without vanishing identically, and for $\beta < 2$, $\beta \neq 1$, it has noninteger growth order $2-\beta$. It is analytic away from zero. None of its derivative orders can therefore vanish on an open interval. The polynomial-test theorem proves the higher Zolotarev failures, even for integer powers with a smooth density. At $\beta = 2$ the usual Gaussian polynomial inversion instead lowers degree by one and yields the familiar three bands.

For a smooth test, expand the numerator of (1.4) near zero and multiply by $e^{\lambda|x|^\beta}$. Generically its leading nonsmooth term is $\lambda f_h(0)|x|^\beta$; the next terms have higher order. For noninteger β, derivative order $\lceil\beta\rceil$ can be unbounded, although the derivative is locally integrable at that order. For odd integer β, derivative order β has jump $2\lambda\beta! f_h(0)$ and the next distributional derivative has an atom. For even integers the potential and these terms are smooth. These statements are additional to the jump of an indicator test at its own threshold. They justify the derivative conventions used in the tables.

3.3 Kolmogorov and bounded-test envelopes

The indicator formula follows by splitting (1.4) at its threshold. Letting the threshold approach x from either side gives K_n. For bounded tests, the order-zero signed measure has positive and negative masses both equal to A. At order one the continuous part in (2.2) has variation $F|R'| + Q|S'|$, and the atom has mass a. Its total mass is zero. Half its total variation is the norm over oscillation-one functions, proving B_1. Smooth approximations to the signs, with a shrinking adjustment at x, prove sharpness.

3.4 Zolotarev and Wasserstein: extremisers and correction terms

Integration by parts in (1.4) gives (2.3); the boundary terms vanish by the finite first moment. Its kernel variation gives T, W_1, W_2. The polynomial-test theorem gives the corresponding higher-order assertion, including every required cancellation. For any nonnegative weight, exchanging the two suprema over tests and points proves that weighting an exact envelope gives the exact weighted factor, including when the result is infinite.

3.5 Fortet–Mourier refinements

For a prescribed value $h(x) = c$, the function in (1.18) is the largest admissible value to the left and the smallest to the right. The two kernel signs therefore prove the formula. For (1.19), the continuous measure is nonpositive and its negative mass is a. The extremiser is $h(t) = \max\{-B, B - L|t-x|\}$, giving the displayed truncated-cone integral. The upper bounds follow by intersecting the bounded and Lipschitz classes. Differentiating the equation once gives (1.17), so it does not need a first moment.

4 Why the missing bounds cannot hold

4.1 Polynomial freedom at low derivative orders

Whenever a polynomial of degree less than r has a nonzero solution derivative of order n, arbitrary multiples invalidate an H_r-only bound. Moment matching for a Zolotarev metric cancels the polynomial's contribution to the difference of expectations, not its contribution to an individual Stein solution. Anchoring lower test derivatives is a legitimate different class; it does not justify deleting correction terms for the unanchored class used here.

4.2 Local smooth counterexamples at high derivative orders

Fix a regular interior point x_0 and choose $h_\varepsilon(t) = \varepsilon^r \psi((t-x_0)/\varepsilon)$ with ψ smooth and compactly supported and $\|\psi^{(r)}\|_\infty \le 1$. Differentiating the equation $r+1$ times exhibits the leading term $a(x_0)h_\varepsilon^{(r+1)}(x_0)$ in $f^{(r+2)}(x_0)$. All other terms have smaller order in ε^{-1}. Choosing a nonzero jet at x_0 proves divergence. The same construction with the relevant higher jet works for every $n \ge r+2$. The case $r = 0$ proves the bounded-test obstruction. For $r = 1$ these bumps also

belong to a fixed rectangular Fortet–Mourier class for sufficiently small ε. A weight positive at x_0 does not remove the failure.

4.3 Boundary divergence is a different obstruction

A finite pointwise envelope with infinite spatial supremum can often be repaired by a weight. This is distinct from failure at a fixed interior point, from a missing target moment, and from absence of a classical derivative at a nonsmooth point of the density. The tables distinguish these four issues; their entries must not be interchanged.

5 Evaluating and using the bounds

5.1 Numerical shape constants and explicit reference values

For $x \geq 0$, the density and upper tail are evaluated through

$$Q(x) = \frac{\Gamma(1/\beta, \lambda x^\beta)}{2\Gamma(1/\beta)}, \qquad \mathbb{E}|Z|^r = \lambda^{-r/\beta}\frac{\Gamma((r+1)/\beta)}{\Gamma(1/\beta)} \quad (r > -1).$$

Use reflection for $x < 0$. The integrated tails at order two are $P_2 = xF + \tau p$ and $Q_2 = -xQ + \tau p$. Higher positive truncated moments can be expanded into incomplete-Gamma integrals, or evaluated directly without cancellation.

At $\beta = 3/2$, $\lambda = 1$, the exact Kolmogorov factor is evaluated by the single incomplete-Gamma root above. The origin value alone is strictly too small. In contrast the first Kolmogorov factor is exactly 1. The Wasserstein order-zero envelope diverges in the tails, while its order-two envelope diverges at the origin. The two failures need weights at different locations, despite occurring for the same target.

Scale. Put $L = \lambda^{-1/\beta}$ and let Z_1 denote the law with $\lambda = 1$. For $Y = LZ_1$ the density-equation solution obeys $g_H^{(n)}(Lx) = L^{1-n} f_{\widetilde{H}}^{(n)}(x)$, where $\widetilde{H}(x) = H(Lx)$. A factor from test order r to solution order n acquires L^{r+1-n}. For bounded and indicator tests use $r = 0$. The weights defined through $\rho = |x|/L$ have the same dimensionless profile and the same scaling rule.

5.2 Stable evaluation and scaling

Compute the smaller tail directly rather than subtracting it from one. Evaluate positive truncated moments by quadrature or special functions; subtractive recurrences are best used away from extreme tails. Use logarithms for small densities. Include analytic endpoint limits and exceptional-point asymptotics when maximising an envelope. A grid or a local optimiser alone does not certify a global maximum. The exact supremum definitions remain valid independently of numerical optimisation; rounded numerical values, where displayed, are illustrations.

5.3 How the weights enter approximation arguments

Write the equation as $wf' + \gamma f = h - \mathbb{E}h(Z)$. Suppose a comparison variable satisfies $\mathbb{E}[-\gamma(X)f(X)] = \mathbb{E}[T_X f'(X)] + \mathbb{E}[R_X f(X)]$ for the solutions under consideration, with all expectations finite. Then

$$\mathbb{E}h(X) - \mathbb{E}h(Z) = \mathbb{E}[(w(X) - T_X)f_h'(X)] - \mathbb{E}[R_X f_h(X)].$$

For a test class with envelopes E_0, E_1, this gives

$$d_{\mathcal{H}}(X, Z) \le \mathbb{E}[|w(X) - T_X|E_1(X)] + \mathbb{E}[|R_X|E_0(X)].$$

One may replace $E_1(X)$ by an ordinary factor or by $\mathfrak{C}_\omega[E_1]/\omega(X)$ where $\omega(X) > 0$. The pointwise version often retains information lost by either uniform factor. At exceptional points use the limiting equation only when it is defined for the chosen tests and when the assumed comparison identity includes that convention.

Relation to the literature and scope. The kernel construction follows the framework of Rapin and Swan [1]; the proofs above make the formulas used here self-contained. The iterative approach of Döbler, Gaunt and Vollmer [3] provides a complementary way to retain lower derivative controls. Exact envelope formulas, their one-dimensional supremum definitions, and convenient upper bounds are distinguished throughout. No blanket claim of numerical improvement over every previously published factor is made.

References

[1] F. Rapin and Y. Swan (2026). *Representations of solutions to Stein equations and their derivatives*. arXiv:2609.17881.

[2] P. Eichelsbacher (2026). *Stein Solution Factors via Mills Ratios: Affine Birth Rates and Symmetric Potential Distributions*. arXiv:2609.25380.

[3] C. Döbler, R. E. Gaunt and S. J. Vollmer (2017). An iterative technique for bounding derivatives of solutions of Stein equations. *Electronic Journal of Probability* **22**, paper 96. arXiv:1510.02623.