Improved explicit Gamma Stein factors from a maximum-point argument

Comparison and proposed addendum to the Gamma companion

26 September 2026

Abstract

The maximum-point method in Eichelsbacher's Mills-ratio paper suggests a useful extension to the integrated tails in our Gamma manuscript. We derive elementary bounds for the ordinary and weighted diagonal factors at every derivative order. Both factors are evaluated exactly when the shape is at most the derivative order plus one. Above this threshold, the bounds retain the correct large-shape orders and substantially improve the elementary estimates in the companion. These are deductions proved here, not Gamma theorems quoted from that paper. We also identify a necessary correction to the Subbotin specialization in the general representation paper. The already sharp Gaussian factors are unchanged.

1 Results ready to use

Let $G \sim \Gamma(a, b)$, with shape $a > 0$ and scale $b > 0$. Keep the companion's equation

$$bxf_h'(x) + (ab - x)f_h(x) = h(x) - \mathbb{E}h(G), \qquad x > 0. \tag{1}$$

At scale one write $p(x) = x^{a-1}e^{-x}/\Gamma(a)$, $P_0 = Q_0 = p$, and

$$P_j(x) = \int_0^x P_{j-1}(t)\,dt, \qquad Q_j(x) = \int_x^\infty Q_{j-1}(t)\,dt \quad (j \geq 1).$$

Thus $P_1 = F$, $Q_1 = 1 - F$. Define, including order zero for the proof,

$$D_n(x) = \frac{2n!P_{n+1}(x)Q_{n+1}(x)}{x^{n+1}p(x)}, \quad n \geq 0.$$

For $n \geq 1$, the manuscript's exact factors are $c_{a,n} = \sup_x D_n(x)$ and $d_{a,n} = \sup_x xD_n(x)$. Also $D_0 = 2\mathcal{A}_{a,1}$, so $u_a = \frac{1}{2}\sup D_0$ and $v_a = \frac{1}{2}\sup xD_0$.

Theorem 1 (Explicit diagonal factors)**.** *For every integer $n \geq 1$,*

$$c_{a,n} \leq C_{a,n} := \begin{cases} \dfrac{2}{a+n}, & 0 < a \leq n+1, \\ \dfrac{2}{\sqrt{(2n+1)(2a-1)}}, & a > n+1, \end{cases} \tag{2}$$

$$d_{a,n} \leq W_{a,n} := \begin{cases} 2, & 0 < a \leq n+1, \\ 2\sqrt{\dfrac{(a+n)^2 - 1}{2\{(2n+1)a - (n+1)\}}}, & a > n+1. \end{cases} \tag{3}$$

Both inequalities are equalities when $0 < a \leq n+1$. For $a > n+1$, these are explicit upper bounds, not asserted optimal constants.

In particular, for the regularity convention in the companion,

$$\|f_h^{(n)}\|_\infty \leq \frac{C_{a,n}}{b}\|h^{(n)}\|_\infty, \qquad \|xf_h^{(n)}\|_\infty \leq W_{a,n}\|h^{(n)}\|_\infty. \tag{4}$$

The x multiplies the derivative; it is not differentiated. For weight bx, multiply the second constant by b.

1.1 Orders zero, one and two

For Kolmogorov and bounded tests the order-zero factors satisfy

$$u_a \le U_a := \begin{cases} 1/a, & a \le 1, \\ 1/\sqrt{2a-1}, & a > 1, \end{cases} \qquad v_a \le V_a := \begin{cases} 1, & a \le 1, \\ \sqrt{(a+1)/2}, & a > 1. \end{cases} \tag{5}$$

Both are exact for $a \le 1$. The first two diagonal orders are

$$C_{a,1} = \begin{cases} 2/(a+1), & a \le 2, \\ 2/\sqrt{3(2a-1)}, & a > 2, \end{cases} \qquad W_{a,1} = \begin{cases} 2, & a \le 2, \\ 2\sqrt{a(a+2)/(6a-4)}, & a > 2, \end{cases}$$

$$C_{a,2} = \begin{cases} 2/(a+2), & a \le 3, \\ 2/\sqrt{5(2a-1)}, & a > 3, \end{cases} \qquad W_{a,2} = \begin{cases} 2, & a \le 3, \\ 2\sqrt{(a+1)(a+3)/(10a-6)}, & a > 3. \end{cases}$$

For fixed $n \ge 1$, the new bounds behave as

$$C_{a,n} \sim \sqrt{\frac{2}{(2n+1)a}}, \qquad W_{a,n} \sim \sqrt{\frac{2a}{2n+1}} \qquad (a \to \infty).$$

These replace the companion's estimates $2/n$ and $2(1+(a-1)_+/n)$, respectively. The new ordinary bound is strictly smaller for every $a > 0$. The weighted bound is equal to the old one for $a \le 1$ and strictly smaller for $a > 1$: above $a = n+1$, one has $W_{a,n}/2 \le (a+n-1)/n$, as follows from $2\{(2n+1)a - (n+1)\} > 4n(n+1)$ and $(a+n+1)/(a+n-1) < 2$.

1.2 Where to insert the improvements

The exact pointwise envelopes in the companion remain the same. Replace only the readily evaluated upper bounds for their suprema.

Test class	Consequence in the results-first tables
Kolmogorov	At order zero, $\|f_z\|_\infty \le U_a/b$ and $\|xf_z\|_\infty \le V_a$. Higher-order indicator bounds are not improved by this argument.
Bounded h	$\|f_h\|_\infty \le U_a \operatorname{osc}(h)/b$ and $\|xf_h\|_\infty \le V_a \operatorname{osc}(h)$. If $\|h\|_\infty \le B$, use $\operatorname{osc}(h) \le 2B$.
Wasserstein	For $\|h'\|_\infty \le L$, $\|f_h'\|_\infty \le LC_{a,1}/b$ and $\|xf_h'\|_\infty \le LW_{a,1}$. The exact bounds $\|f_h\|_\infty \le L$ and $\|xf_h''\|_\infty \le 2L/b$ stay unchanged.
Fortet–Mourier	If $\|h\|_\infty \le B$ and $\|h'\|_\infty \le L$, use $\|f_h\|_\infty \le \min(2BU_a/b, L)$, $\|xf_h\|_\infty \le 2BV_a$, $\|f_h'\|_\infty \le LC_{a,1}/b$, and $\|xf_h'\|_\infty \le \min(2B/b, LW_{a,1})$.
Integer Zolotarev	Under $\|h^{(r)}\|_\infty \le 1$, replace diagonal factors by $C_{a,r}/b$ and $W_{a,r}$. The neighbouring factors $1/r$ and $2/b$ for $f^{(r-1)}$ and $xf^{(r+1)}$ are unchanged.

The impossibility statements for insufficiently smooth tests also remain unchanged. A smaller global upper bound cannot improve an already exact pointwise envelope.

1.3 Size of the improvement

At shape $a = 5$ and scale $b = 1$:

Factor	Previous explicit bound	New explicit bound	Numerical supremum
$c_{5,1}$	2	0.38490	0.38053
$d_{5,1}$	10	2.32048	2.31163
$c_{5,2}$	1	0.29814	0.29754
$d_{5,2}$	6	2.08893	2.08856

Displayed decimals are rounded comparisons; use the formulas for rigorous upper bounds. The last column is numerical optimization, not a certified enclosure. The improvement is to the explicit substitute for the exact supremum, not to that supremum itself.

2 Proof by reciprocal integrated tails

This is a continuous adaptation of the strategy behind the maximum-index argument of [1, Theorem 3.1]: impose the first- and second-order conditions at a maximizer, then eliminate the tail ratios. Our integrated-tail identities allow the argument to be carried out at every Gamma derivative order.

Lemma 1 (Riccati identities)**.** *For $n \geq 0$ set*

$$R(x) = \frac{xP_n(x)}{P_{n+1}(x)}, \qquad S(x) = \frac{xQ_n(x)}{Q_{n+1}(x)}, \qquad A = a + n.$$

Then

$$D_n = \frac{2}{R+S}, \tag{6}$$

$$R' = n + \frac{(A-x)R - R^2}{x}, \qquad S' = -n + \frac{(A-x)S + S^2}{x}, \tag{7}$$

$$(\log D_n)' = \frac{x - A + R - S}{x}. \tag{8}$$

Proof. The Gamma recurrences, for $n \geq 1$, are

$$nP_{n+1} = (x - a - n + 1)P_n + xP_{n-1}, \quad nQ_{n+1} = (a + n - 1 - x)Q_n + xQ_{n-1}.$$

Their cancellation gives, by induction,

$$n!\{P_{n+1}Q_n + Q_{n+1}P_n\} = x^n p.$$

This proves (6). Differentiate the ratios and use the recurrences to get (7); at $n = 0$, use $p'/p = (a-1)/x - 1$ directly. Logarithmic differentiation of D_n proves (8). □

The endpoint limits, directly from the Gamma moments and tail asymptotics, are

$$D_n(0+) = \frac{2}{a+n}, \quad D_n(x) \sim \frac{2}{x}\ (x \to \infty), \quad xD_n(x) \longrightarrow \begin{cases} 0, & x \downarrow 0, \\ 2, & x \to \infty. \end{cases} \tag{9}$$

Thus a supremum strictly above both endpoint limits is attained at an interior maximum. Endpoint limits must be retained: assuming interior attainment would lose the exact small-shape regimes.

2.1 Ordinary factors

At an interior local maximum x_* of D_n, (8) gives $R - S = A - x_*$. Differentiating its numerator and using (7), the second-order condition yields

$$R^2 + S^2 \geq (2n+1)x_* + (A - x_*)^2.$$

Consequently

$$\begin{aligned}\frac{4}{D_n(x_*)^2} = (R+S)^2 &\geq 2(2n+1)x_* + (A-x_*)^2 \\ &= x_*^2 + 2(n+1-a)x_* + (a+n)^2 \\ &= \{x_* - (a-n-1)\}^2 + (2n+1)(2a-1). \end{aligned} \tag{10}$$

For $a \leq n+1$, the second line is at least $(a+n)^2$. For $a > n+1$, the last line is at least $(2n+1)(2a-1) > 0$. The boundary value $2/(a+n)$ also satisfies the resulting upper bound. Equation (9) now proves (2), including exactness in the first regime. The same proof at $n = 0$ proves the assertion about u_a in (5).

2.2 Weighted factors

At an interior local maximum x_* of $xD_n(x)$, stationarity gives $R - S = A - 1 - x_* =: d$. The second-order condition now gives

$$R^2 + S^2 \geq (2n+1)x_* + d(d+1).$$

Therefore

$$\frac{4}{D_n(x_*)^2} \geq x_*^2 - 2cx_* + K, \qquad c = a - n - 1, \quad K = (a+n)^2 - 1. \tag{11}$$

If $n \geq 1$ and $a \leq n+1$, then $c \leq 0$ and $K > 0$, so $x_* D_n(x_*) \leq 2$. The endpoint limit at infinity makes the factor exactly 2.

If $a > n+1$, then $c > 0$ and

$$K - c^2 = 2\{(2n+1)a - (n+1)\} > 0.$$

Optimizing the elementary rational function gives

$$\sup_{x>0} \frac{x^2}{x^2 - 2cx + K} = \frac{K}{K - c^2}, \qquad x_{\text{rational max}} = K/c.$$

Together with (11) and the endpoint values, this proves (3). This last rational maximizer need not be the maximizer of the original envelope.

The same reasoning at $n = 0$ proves $v_a \leq \sqrt{(a+1)/2}$ for $a > 1$. For $0 < a \leq 1$, use instead

$$\frac{Q_1(x)}{p(x)} = \int_0^\infty (1 + t/x)^{a-1} e^{-t}\, dt \leq 1.$$

Hence $x\mathcal{A}_{a,1}(x) = F(x)Q_1(x)/p(x) \leq 1$, with limit one at infinity. This proves the remaining case of (5).

Scope of the inequalities. Equations (10) and (11) are conditions at local maxima, not pointwise envelope inequalities for arbitrary x. Their use here is justified by the endpoint analysis. The resulting global factors are valid everywhere.

3 What comes from the cited paper, and what is derived here

The cited paper treats affine birth rates and symmetric density operators [1]. Its negative-binomial Corollary 4.5 is consistent with the u_a bound above under Gamma scaling: if $N \sim \mathrm{NB}(a, p)$ and $p \downarrow 0$, then $pN \Rightarrow \Gamma(a, 1)$, and for $k \sim x/p$, the discrete envelope tends to $F(x)(1 - F(x))/(xp(x))$ (with the final $p(x)$ denoting the Gamma density). The bound $\{(1 - p)(2a - 1)\}^{-1/2}$ tends to $(2a - 1)^{-1/2}$. The direct proof above avoids a limiting argument and covers $a < 1$ as well.

The all-order ordinary and weighted Gamma formulas (2)–(3) are established here using the companion's integrated tails; they should not be attributed as stated theorems of [1]. No claim of priority over the wider literature is made. The exact small-shape ordinary factor is already reported in [2, Eq. (3.14)]; the argument above supplies a short proof of its supremum value without requiring global monotonicity of the envelope.

4 A necessary Subbotin correction

This concerns the broader representation paper [2], not the Gaussian or Gamma companion. Its Section 3.2 starts with $\beta > 1$, but Eq. (3.24) assigns the order-zero Kolmogorov factor $1/(4p_\beta(0))$ throughout that range. That value is correct for $\beta \geq 2$; it is too small when $1 < \beta < 2$. The bounded-test factor immediately before Eq. (3.29) requires the same correction. This agrees with [1, Proposition 8.5].

Here is a direct check of the obstruction. For an even density $p(x) = ce^{-\lambda|x|^\beta}$ with $1 < \beta < 2$, as $x \downarrow 0$,

$$\frac{F(x)(1 - F(x))}{p(x)} = \frac{1}{4c} + \frac{\lambda}{4c}x^\beta - cx^2 + o(x^2).$$

Since x^β dominates x^2, the envelope increases above its value at zero. Thus the claimed centre maximum cannot hold in this regime.

For an implementable replacement put $s = 1/\beta$ and $P_s(u) = \gamma(s, u)/\Gamma(s)$. The exact factor is

$$C_K(\beta, \lambda) = \frac{\Gamma(s)}{2\beta\lambda^s} \max_{u \geq 0} \left[e^u \{1 - P_s(u)^2\} \right]. \tag{12}$$

For $1 < \beta < 2$, the maximizer is the unique positive root of

$$2P_s(u)P_s'(u) = 1 - P_s(u)^2,$$

by [1, Proposition 8.5]. Use $\lambda = 1/[\beta(\beta - 1)]$ for [2]. The factor for $\|h\|_\infty \leq 1$ is $2C_K$. Numerically, the ratio of the true Kolmogorov factor to the centre value is approximately 1.41334 at $\beta = 1.2$, 1.09952 at $\beta = 1.5$, and 1.00536 at $\beta = 1.8$.

Gaussian and other factors. The Gaussian case $\beta = 2$ retains the sharp Kolmogorov constants $\sqrt{2\pi}/4$ and 1 at orders zero and one. The cited paper supplies no reduction of these constants. Its drift-weighted indicator bound can sharpen a statement of "≤ 1" to an equality of the global supremum for the Subbotin family. Its continuous results concern indicator tests and symmetric potentials; they do not directly improve our higher-order Wasserstein, Fortet–Mourier or Zolotarev factors. Those Gamma improvements instead follow from the argument proved above.

5 Recommended manuscript changes and checks

1. Replace the elementary Gamma estimates by (2)–(3); add (5) to the order-zero tables. Retain the exact supremum definitions for users who want tighter numerical factors.

2. Insert Lemma 1 and the maximum-point proof, explicitly treating the boundary suprema. Cross-reference the resulting constants across all five test classes.

3. Restrict the Subbotin centre-maximum formula to $\beta \geq 2$; use (12) below that threshold, and correct the corresponding bounded-test statement.

The formulas were checked numerically on 1,764 combinations of shapes $0.25, 0.5, 1, 1.5, 2, 3, 5, 10, 50$, orders 0 through 6, and positive evaluation points. Numerical optimization supplied the comparison tables and Subbotin ratios. The proofs, rather than those numerical checks, establish the inequalities. This addendum leaves the existing manuscript files unchanged so that the deductions and corrections can be reviewed together.

References

[1] P. Eichelsbacher. *Stein Solution Factors via Mills Ratios: Affine Birth Rates and Symmetric Potential Distributions*. arXiv:2609.25380v1, 2026. https://arxiv.org/abs/2609.25380.

[2] F. Rapin and Y. Swan. *Representations of solutions to Stein equations and their derivatives*. arXiv:2609.17881v1, 2026. https://arxiv.org/abs/2609.17881.

[3] G. Bailly, F. Rapin, Y. Swan and R. von Sachs. *Explicit Gamma-Stein Bounds for Satterthwaite's Approximation*. arXiv:2609.17880v1, 2026. https://arxiv.org/abs/2609.17880.