Inverse Gamma Stein envelopes and factors: ordinary and weighted bounds by test class

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Abstract

We organise pointwise envelopes and uniform factors for the Inverse Gamma Stein equation by test class: Kolmogorov, bounded measurable, Wasserstein, Fortet–Mourier, and integer-order Zolotarev. The first section gives the bounds at derivative orders zero, one, two, and general order, with separate ordinary and weighted factors. Integrated tails and weighted Mills ratios give exact envelopes at the available single-seminorm orders. Polynomial and localised smooth examples explain unavailable bounds. The moment threshold is stated separately for every smooth test class. The last admissible moment band is treated by absolute kernel masses, without assuming unavailable higher moments. Exact supremum formulas are distinguished from explicit upper bounds and numerical illustrations. The structure and conventions follow the Gaussian, Gamma and Beta companion papers.

Keywords: Stein equation; Stein factors; weighted derivative bounds; kernel representation; probability metrics.

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1 Results at a glance and by test class

1.1 Equation, normalisation, and the meaning of a weighted factor

Let $Z \sim \mathrm{InvGamma}(\alpha, b)$, $\alpha, b > 0$, with $I = (\ell, u) = (0, \infty)$ and

$$p(x) = \frac{b^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-b/x}.$$

We use the equation

$$x^2 f_h'(x) + \{b - (\alpha - 1)x\} f_h(x) = h(x) - \mathbb{E}h(Z). \tag{1.1}$$

Thus $w = x^2$, $\gamma = b - (\alpha - 1)x$, and put $m = \alpha$. For $\alpha > 1$, $\mu = b/(\alpha - 1)$ and $\tau = w/(\alpha - 1)$. If g solves $\tau g' + (\mu - x)g = h - \mathbb{E}h(Z)$, then $g = (\alpha - 1)f$. Multiply ordinary factors by $\alpha - 1$ to convert; $\tau g^{(n)} = w f^{(n)}$. The main weighted column uses $\omega = w = x^2$.

For $\mathbb{E}|h(Z)| < \infty$, the canonical solution is

$$f_h(x) = \frac{\int_\ell^x \{h(t) - \mathbb{E}h(Z)\} p(t)\, dt}{w(x)p(x)} = -\frac{\int_x^u \{h(t) - \mathbb{E}h(Z)\} p(t)\, dt}{w(x)p(x)}. \tag{1.2}$$

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The endpoint flux wpf_h vanishes at both ends. This specifies the solution; adding a constant to h leaves f_h unchanged. Write $F(x) = \Pr(Z \le x)$ and $Q = 1 - F$. All norms are over the support, with the one-sided conventions specified below at exceptional points. For bounded tests put $H_0 = \operatorname{osc}(h) = \sup h - \inf h$. For $r \ge 1$, $H_r = \|h^{(r)}\|_\infty < \infty$ means $h \in C^{r-1}$, with locally absolutely continuous $(r-1)$st derivative and essentially bounded rth derivative. Integrability is a separate requirement.

An *envelope* bounds $|f_h^{(n)}(x)|$ over the indicated normalised class. An exact envelope is the pointwise class supremum, with limiting extremisers allowed. Its ordinary factor is its spatial supremum; a weighted factor is the supremum after multiplication by a specified nonnegative weight ω. Thus $\omega f_h^{(n)}$ is a derivative multiplied by a weight, not $(\omega f_h)^{(n)}$. Throughout the tables, use the explicit evaluation rules

$$\mathfrak{C}[E] := \sup_{x \in I} E(x), \qquad \mathfrak{C}_\omega[E] := \sup_{x \in I} \omega(x) E(x). \tag{1.3}$$

These are one-dimensional supremum formulas for the factors, not unspecified constants. An entry ∞ means that a pointwise envelope exists but has infinite spatial supremum. "None" denotes failure of a finite class envelope on an open set of interior points. "Not defined" means that the full test class contains nonintegrable functions.

The quantities needed in the tables. Set $P_0 = Q_0 = p$ and, whenever finite,

$$P_j(x) = \int_\ell^x P_{j-1}(t)\,dt, \qquad Q_j(x) = \int_x^u Q_{j-1}(t)\,dt \quad (j \ge 1). \tag{1.4}$$

Thus $P_1 = F$, $Q_1 = Q$. Define

$$a(x) = 1/w(x), \qquad d(x) = -\frac{(wp)'(x)}{w(x)p(x)},$$
$$S(x) = \frac{F(x)}{w(x)p(x)}, \qquad R(x) = \frac{Q(x)}{w(x)p(x)}, \tag{1.5}$$
$$A(x) = \frac{F(x)Q(x)}{w(x)p(x)}, \qquad K_n(x) = \max\{Q(x)|S^{(n)}(x)|, F(x)|R^{(n)}(x)|\}, \tag{1.6}$$

$$B_1(x) = \frac{a(x) + F(x)|R'(x)| + Q(x)|S'(x)|}{2}. \tag{1.7}$$

In particular $K_0 = A$. When $\mathbb{E}|Z| < \infty$, also set

$$T(x) = R(x)P_2(x) + S(x)Q_2(x) = \tau(x)/w(x), \tag{1.8}$$
$$W_1(x) = |R'(x)|P_2(x) + |S'(x)|Q_2(x), \tag{1.9}$$
$$W_2(x) = a(x) + |R''(x)|P_2(x) + |S''(x)|Q_2(x). \tag{1.10}$$

Here $\tau p = \int_\ell^x (\mu - t)p(t)\,dt$ and $\mu = \mathbb{E}Z$. No higher-order moment is needed for W_2. For direct calculation, only two differentiations are needed:

$$S' = a + dS, \qquad R' = -a + dR,$$
$$S'' = a' + da + (d' + d^2)S, \qquad R'' = -a' - da + (d' + d^2)R. \tag{1.11}$$

For $r \ge 1$ with finite rth absolute moment, write

$$E_{n,r}(x) = |R^{(n)}(x)|P_{r+1}(x) + |S^{(n)}(x)|Q_{r+1}(x) + \mathbf{1}_{\{n=r+1\}}a(x), \quad 0 \le n \le r+1. \tag{1.12}$$

This last envelope applies only when the polynomial cancellation condition in the Zolotarev subsection holds. In particular it must not automatically be called a diagonal Zolotarev envelope.

Moment range and the three neighbouring orders. The target has a finite positive absolute moment of order r exactly when $r < m$. Bounded and Kolmogorov tables are valid for every parameter specified above. The full Wasserstein class requires $m > 1$; the full integer Zolotarev class of order r requires $r < m$. For $n < m$ set

$$q_n = n(m-n), \qquad \kappa_n = \prod_{j=1}^{n} q_j, \qquad D_n = \frac{2\kappa_n P_{n+1} Q_{n+1}}{w^{n+1} p}, \quad \kappa_0 = 1. \tag{1.13}$$

In particular $D_0 = 2A$. Put $c_n = \mathfrak{C}[D_n]$, $e_r = \mathfrak{C}[E_{r+1,r}]$, $j_r = \mathfrak{C}[wE_{r+1,r}]$. For $m > 1$, $T = 1/(m-1)$, $W_1 = D_1$ and $B_1 = 1/w$. For $m > 2$, $W_2 = 2/w$. If $1 < m \le 2$, the exact formula W_2 in (1.10) remains valid without a second moment; do not substitute $2/w$ without checking its signs.

Normalised class	Order 0	Order 1	Order 2
Kolmogorov	A	K_1	K_2 (piecewise)
Bounded, $H_0 \le 1$	A	B_1	none
Wasserstein, $m > 1$	$1/(m-1)$	D_1	W_2
Fortet–Mourier, $B = L = 1$	V_0 (upper)	V_1 (upper)	V_2 (upper)
Zolotarev $r = 2$, $m > 2$	none	$1/[2(m-2)]$	D_2

This overview lists pointwise envelopes. Apply (1.3) for ordinary and weighted factors; their finiteness and useful simplifications are recorded next, before the separate class tables.

Ordinary factors and the main weight. At zero, $A(0+) = 1/b$ and $D_n(0+) = 2/b$ for $n < \alpha$. At infinity, $A(x) \sim 1/(\alpha x)$ and $D_n(x) \sim 2/(\alpha x)$. Thus $u_0 := \mathfrak{C}[A]$ and $c_n := \mathfrak{C}[D_n]$ are finite, but their x^2-weighted factors are infinite. For $n \ge 1$,

$$K_n(x) \sim b^{n-1} x^{-2n} \quad (x \downarrow 0), \qquad K_n(x) \asymp x^{-n-1} \quad (x \to \infty).$$

All positive-order ordinary Kolmogorov factors are infinite. The x^2-weighted factor is finite at order one and infinite for $n \ge 2$. For $\alpha > 1$, $B_1 = 1/x^2$ and the weighted factor is exactly 1. For $\alpha \le 1$, the general formula for B_1 still has infinite ordinary factor and finite x^2-weighted factor. For $1 \le r < \alpha$, $E_{r+1,r} \sim 2/x^2$ at zero and is of order x^{-2} at infinity. Consequently $e_r = \infty$ and $j_r := \mathfrak{C}[x^2 E_{r+1,r}] < \infty$; if $r + 1 < \alpha$, $j_r = 2$.

Other weights. For a power weight x^q, the exact admissible ranges are $0 \le q \le 1$ for A, D_r, $q = 0$ for a nonzero constant, $q = 2$ for $E_{r+1,r}$, and $2n \le q \le n + 1$ for K_n ($n \ge 1$). The last range is empty for $n \ge 2$. A useful two-endpoint replacement is

$$\omega_n(x) = \frac{x^{2n}}{(1 + x/b)^{n-1}}, \qquad n \ge 1,$$

for which $\mathfrak{C}[\omega_n K_n] < \infty$ at every order.

1.2 Kolmogorov tests: piecewise derivatives and the required weights

Let $h_z(t) = \mathbf{1}_{\{t \le z\}}$ and $f_z = f_{h_z}$.

Theorem 1.1 (Kolmogorov envelopes and factors)**.** *For every order for which the branch derivative is defined,*

$$f_z^{(n)}(x) = \begin{cases} Q(z)S^{(n)}(x), & x < z, \\ F(z)R^{(n)}(x), & x > z. \end{cases}$$

The exact pointwise envelopes and factors are

Order	Exact envelope	Ordinary factor	ω-weighted factor
0	A	$\mathfrak{C}[A]$	$\mathfrak{C}_\omega[A]$
1	K_1	$\mathfrak{C}[K_1]$	$\mathfrak{C}_\omega[K_1]$
2	K_2	$\mathfrak{C}[K_2]$	$\mathfrak{C}_\omega[K_2]$
$n \ge 0$	K_n	$\mathfrak{C}[K_n]$	$\mathfrak{C}_\omega[K_n]$

The solution is continuous at the threshold and $f_z'(z+) - f_z'(z-) = -a(z)$ at regular points. Derivatives in this table are branch derivatives; distributional derivatives can contain atoms.

1.3 Bounded measurable tests

The class $\operatorname{osc}(h) \le 1$ induces total variation in the convention $d_{\mathrm{TV}}(X, Z) = \sup_B |\Pr(X \in B) - \Pr(Z \in B)|$. The class $\|h\|_\infty \le 1$ induces twice that distance.

Theorem 1.2 (Bounded-test envelopes and factors)**.** *Let $H_0 = \operatorname{osc}(h) < \infty$. The following finite bounds are sharp:*

Order	Pointwise bound	Ordinary factor	ω-weighted factor
0	$A(x)H_0$	$\mathfrak{C}[A]$	$\mathfrak{C}_\omega[A]$
1	$B_1(x)H_0$ a.e.	$\mathfrak{C}[B_1]$	$\mathfrak{C}_\omega[B_1]$
2	no finite class envelope	none	none
$n \ge 2$	no finite class envelope	none	none

Factors in the last two columns multiply H_0. If $\|h\|_\infty \le B$, replace H_0 by $2B$; the resulting single-norm bounds are also sharp.

The higher-order negative statements hold even for smooth bounded tests. A weight which is positive near a regular interior point cannot restore these missing bounds. This differs from Kolmogorov tests, whose special piecewise form permits derivatives of all orders on smooth branches.

1.4 Wasserstein tests

Let $L = \mathrm{Lip}(h) = \|h'\|_\infty$. Assume the target has a finite first moment. The unit class induces Wasserstein distance when the compared variable also has a finite first moment.

Theorem 1.3 (Wasserstein envelopes and factors)**.** *On each regular open interval the solution belongs to C^1, its first derivative is locally absolutely continuous, and*

Order	Exact envelope times L	Ordinary factor	ω-weighted factor
0	$T(x)L$	$\mathfrak{C}[T]$	$\mathfrak{C}_\omega[T]$
1	$W_1(x)L$	$\mathfrak{C}[W_1]$	$\mathfrak{C}_\omega[W_1]$
2	$W_2(x)L$ a.e.	$\mathfrak{C}[W_2]$	$\mathfrak{C}_\omega[W_2]$
$n \geq 3$	no finite class envelope	none	none

The finite envelopes and factors are exact; factors multiply L.

The order-zero extremiser is $h(t) = t$, giving $f_h = -T$. For order one choose h' to match the signs of the two kernel pieces. For order two its value at the evaluation point can additionally be chosen on a shrinking interval, giving the extra term a in W_2.

1.5 Fortet–Mourier tests: size and Lipschitz constraints together

First use the rectangular convention $\mathcal{H}_{B,L} = \{h : \|h\|_\infty \leq B,\ \mathrm{Lip}(h) \leq L\}$. The usual rectangular unit class has $B = L = 1$. When the target has a first moment put

$$V_0 = \min\{2BA, LT\}, \quad V_1 = \min\{2BB_1, LW_1\}, \quad V_2 = LW_2.$$

The following table gives upper bounds; the first two minima are not asserted to be exact under both constraints:

Order	Pointwise upper bound	Ordinary upper factor	Weighted upper factor
0	V_0	$\mathfrak{C}[V_0]$	$\mathfrak{C}_\omega[V_0]$
1	V_1	$\mathfrak{C}[V_1]$	$\mathfrak{C}_\omega[V_1]$
2	V_2 a.e.	$\mathfrak{C}[V_2]$	$\mathfrak{C}_\omega[V_2]$
$n \geq 3$	no finite class envelope	none	none

Without a first moment, use $V_0 = 2BA$, $V_1 = 2BB_1$ and the valid replacement

$$V_2 = aL + 2B|a' + da| + 2BA|d' + d^2|. \tag{1.14}$$

This replacement is also available with a first moment: take the minimum of the two order-two upper bounds.

Exact refinements at orders zero and one. At order zero an exact one-parameter formula is

$$\max_{-B \leq c \leq B} \left\{ R(x) \int_\ell^x p(t)\, \mathrm{clip}_{[-B,B]}(c + L(x-t))\, dt - S(x) \int_x^u p(t)\, \mathrm{clip}_{[-B,B]}(c + L(x-t))\, dt \right\}. \tag{1.15}$$

When $R' \leq 0 \leq S'$, the exact first-order envelope is

$$-R'(x) \int_\ell^x p(t) \min\{2B, L|t-x|\}\, dt + S'(x) \int_x^u p(t) \min\{2B, L|t-x|\}\, dt. \tag{1.16}$$

Its factors are obtained by the same supremum rules (1.3). When these signs fail, the exact first-order envelope is the bounded Lipschitz norm of the explicitly known zero-mass measure

$$R'(x)p(t)\mathbf{1}_{t<x}\,dt - S'(x)p(t)\mathbf{1}_{t>x}\,dt + a(x)\delta_x(dt);$$

the preceding table remains an immediately usable upper bound. For the alternative sum norm $\|h\|_\infty + \mathrm{Lip}(h) \le 1$, take the supremum of the exact rectangular envelope over $B + L \le 1$. It is a different normalisation; rectangular factors must not be called sum-norm sharp.

1.6 Integer-order Zolotarev tests

For an integer $1 \le r < m$, use the unanchored class $\mathcal{Z}_r = \{h : H_r \le 1\}$. Lower derivatives are not bounded or anchored. The associated ideal metric between laws with finite rth absolute moments is finite precisely when moments through degree $r-1$ match. For $r = 1$ it is Wasserstein. For $r \ge m$ the full class is not target integrable; restricting it to integrable tests is a different question.

Theorem 1.4 (All-order Zolotarev classification)**.** *For $r < m$ the exact entries are*

Solution order	Exact envelope	Ordinary factor	w-weighted factor
$n < r-1$	no finite class envelope	none	none
$r-1$	$1/q_r$	$1/q_r$	∞
r	D_r	c_r	∞
$r+1$	$E_{r+1,r}$	e_r	j_r
$n \ge r+2$	no finite class envelope	none	none

For $r+1 < m$, the upper neighbouring envelope is exactly $2/w$ and $j_r = 2$. For $r < m \le r+1$ retain the absolute-value expression (1.12). It is an exact, finite pointwise formula, not a claim that the last neighbouring kernel has one sign. For a general H_r, multiply the finite envelopes and factors by H_r.

The requested low orders can be read off without reindexing:

Class (moment condition)	Order 0	Order 1	Order 2
$\mathcal{Z}_1$, $m > 1$	$1/(m-1)$	D_1	W_2
$\mathcal{Z}_2$, $m > 2$	none	$1/[2(m-2)]$	D_2
$\mathcal{Z}_3$, $m > 3$	none	none	$1/[3(m-3)]$
$\mathcal{Z}_r$, $4 \le r < m$	none	none	none

All these finite entries are sharp. For example $h(t) = t^r/r!$ gives $f_h^{(r-1)} = -1/q_r$. The general-order row remains usable even in the last admissible moment band, where the simpler sign formula may fail.

1.7 Combining smoothness controls and choosing other weights

Each term below is available under its own regularity and integrability assumption; omit terms for which the indicated control is unavailable:

$$\begin{aligned} |f_h(x)| &\le \min\{A(x)H_0, T(x)H_1\}, \\ |f_h'(x)| &\le \min\{B_1(x)H_0, W_1(x)H_1\}. \end{aligned} \tag{1.17}$$

An available Zolotarev entry $E_{n,r}H_r$ can be added to the minimum for solution order n. Multiply an envelope by the desired weight before taking its supremum. Each single-seminorm envelope is sharp; sharpness of its minimum under simultaneous constraints is not asserted.

There is also an explicit all-order bound retaining lower controls. Define $A_1 = d$, $b_{1,0} = a$ and recursively, with missing coefficients zero,

$$\begin{aligned} A_{n+1} &= A_n' + dA_n, \\ b_{n+1,0} &= b_{n,0}' + aA_n, \qquad & b_{n+1,j} &= b_{n,j}' + b_{n,j-1} \quad (1 \le j \le n). \end{aligned} \tag{1.18}$$

Then, at every regular point,

$$f_h^{(n)} = A_n f_h + b_{n,0}(h - \mathbb{E}h(Z)) + \sum_{j=1}^{n-1} b_{n,j} h^{(j)}. \tag{1.19}$$

Consequently a valid envelope is

$$|A_n| A H_0 + |b_{n,0}| H_0 + \sum_{j=1}^{n-1} |b_{n,j}| H_j.$$

For Lipschitz, possibly unbounded tests replace its first two terms by $\{|A_n|T + |b_{n,0}|\mathbb{E}|Z - x|\}H_1$. These are upper bounds, not exact single-seminorm factors.

2 The kernel calculus

2.1 Integrated tails and weighted Mills ratios

The tails have the positive-integral representations

$$P_j(x) = \frac{\mathbb{E}(x - Z)_+^{j-1}}{(j-1)!}, \qquad Q_j(x) = \frac{\mathbb{E}(Z - x)_+^{j-1}}{(j-1)!} \quad (j \ge 1).$$

They are finite whenever the $(j-1)$st absolute moment exists. The derivatives of S, R can be computed without numerical differentiation of a quotient: start from (1.11) and differentiate recursively. No assumption of complete monotonicity is made when it has not been proved.

2.2 The array, its signs, and its three special representations

For $i, j \ge 0$ and regular x, define

$$K^{i,j}(x,t) = \frac{(-1)^j R^{(i)}(x) P_j(t) \mathbf{1}_{t \le x} - S^{(i)}(x) Q_j(t) \mathbf{1}_{t > x}}{p(t)}. \tag{2.1}$$

For bounded tests,

$$\begin{aligned} f_h &= \mathbb{E}[K^{0,0}(x, Z) h(Z)], \\ f_h' &= \mathbb{E}[K^{1,0}(x, Z) h(Z)] + a(x) h(x). \end{aligned} \tag{2.2}$$

The first signed kernel has zero mean; the second has mean $-a$. For Lipschitz tests with a finite first target moment,

$$\begin{aligned} f_h &= -R \int_\ell^x F(t) h'(t)\, dt - S \int_x^u Q(t) h'(t)\, dt, \\ f_h' &= -R' \int_\ell^x F(t) h'(t)\, dt - S' \int_x^u Q(t) h'(t)\, dt, \\ f_h'' &= -R'' \int_\ell^x F(t) h'(t)\, dt - S'' \int_x^u Q(t) h'(t)\, dt + a h'(x). \end{aligned} \tag{2.3}$$

In particular $\mathbb{E}|K^{n,r}(x, Z)| = |R^{(n)}| P_{r+1} + |S^{(n)}| Q_{r+1}$. The last pointwise term in (2.3) must be retained.

2.3 Representations at every derivative order

Put $m_j(t) = t^j/j!$ and $\pi_{n,j}(x) = f_{m_j}^{(n)}(x)$.

Theorem 2.1 (The polynomial test for a single-seminorm bound)**.** *Suppose the rth absolute moment is finite and x is a regular point. For $0 \le n \le r+1$, a finite exact envelope depending only on H_r exists at x if and only if*

$$\pi_{n,j}(x) = 0 \quad (1 \le j < r). \tag{2.4}$$

When this holds, the envelope for $H_r \le 1$ is exactly $E_{n,r}(x)$. For $n \ge r+2$, no finite envelope exists at a regular point where $a > 0$.

Proof. Necessity follows by multiplying any offending polynomial by an arbitrary constant: its rth derivative vanishes. For sufficiency use the integral Taylor formula, subtracting its polynomial of degree $r-1$. The functional $h \mapsto f_h^{(n)}(x)$ annihilates that polynomial by (2.4). For a truncated power $(t-v)_+^{r-1}/(r-1)!$, its image is the kernel $(-1)^r R^{(n)}(x)P_r(v)$ when $v < x$ and $-S^{(n)}(x)Q_r(v)$ when $v > x$. At $n = r+1$ there is in addition $a(x)\delta_x(dv)$. The total variation is precisely (1.12). The integral Taylor identity can first be applied to compactly supported highest derivatives and then passed to the limit; the finite rth moment provides domination of the tails. Choosing $h^{(r)}$ to match the density signs, and choosing its value on a shrinking interval at x independently of the rest, proves sharpness. Smoothing yields the same suprema. The high-order failure follows from the local construction in Section 4. □

Thus finite integrated tails alone do not imply a valid Zolotarev bound. The polynomial test is what detects the correction terms.

3 Proofs of the envelopes, weights, and sharpness

3.1 Endpoint asymptotics and finiteness of the shape constants

At a power-law tail $p(x) \sim Cx^{-m-1}$, positive integration gives

$$Q_{n+1}(x) \sim \frac{Cx^{n-m}}{m(m-1)\cdots(m-n)}, \qquad P_{n+1}(x) \sim \frac{x^n}{n!} \quad (n < m).$$

For a two-sided target apply reflection at the other tail. Substitution into (1.13) proves $D_n \sim 2/(m|x|)$ and $A \sim 1/(m|x|)$. The reciprocal-density branch and the decaying Mills branch give $K_n \asymp |x|^{-n-1}$. This includes integer m: even if the leading polynomial term differentiates to zero, the decaying branch is still present. For $r < m$, substitution directly into (1.12) gives a positive x^{-2} tail for the upper neighbouring envelope. In particular, at $r = 1$, the wW_2 tail limit is 2 for $m \ge 2$ and $4/m$ for $1 < m < 2$. At finite endpoints use $p(x) \asymp x^{a-1}$ for the F case, or the Laplace endpoint expansion $F \sim x^2 p/b$ for inverse Gamma. Repeated positive integration gives the limits and orders stated in Section 1. They also prove every power-weight classification there.

3.2 Integrated-Pearson cancellations and the last moment band

The integrated-Pearson structure [3] has $w'' = 2$ and $(wp)' = \gamma p$ with $\gamma' = 1 - m$. Integration and differentiation yield, whenever $j < m$,

$$\begin{aligned} q_j P_{j+1} &= -\{\gamma + (j-1)w'\}P_j + wP_{j-1}, \\ q_j Q_{j+1} &= \{\gamma + (j-1)w'\}Q_j + wQ_{j-1}. \end{aligned} \tag{3.1}$$

Induction gives for $n < m$

$$S^{(n)} = \frac{\kappa_n P_{n+1}}{w^{n+1}p} > 0, \qquad (-1)^n R^{(n)} = \frac{\kappa_n Q_{n+1}}{w^{n+1}p} > 0.$$

The recurrence also gives the Wronskian $\kappa_n(P_{n+1}Q_n + Q_{n+1}P_n) = w^n p$. For the lower neighbouring kernel its absolute mass is $1/q_r$; for the diagonal it is D_r; when $r+1 < m$, the upper neighbouring kernel is nonpositive, of mass $-1/w$. Its additional pointwise atom therefore gives the exact envelope $2/w$.

The operator maps polynomials of degree $j-1$ to degree j, with nonzero leading coefficient $-(m-j)$. For $j \le r < m$ it therefore has a polynomial canonical inverse of degree $j-1$. The boundary flux of this inverse vanishes since its degree is below the tail index. This proves exactly the cancellations required for $n \ge r-1$. For $r < m \le r+1$ the derivative $S^{(r+1)}$ still exists, but the integrated tail P_{r+2} or Q_{r+2} need not. Apply the polynomial-test theorem directly with the finite P_{r+1}, Q_{r+1} instead. For $n < r-1$, a polynomial of degree $n+1$ gives the stated obstruction.

3.3 Kolmogorov and bounded-test envelopes

The indicator formula follows by splitting (1.2) at its threshold. Letting the threshold approach x from either side gives K_n. For bounded tests, the order-zero signed measure has positive and negative masses both equal to A. At order one the continuous part in (2.2) has variation $F|R'| + Q|S'|$, and the atom has mass a. Its total mass is zero. Half its total variation is the norm over oscillation-one functions, proving B_1. Smooth approximations to the signs, with a shrinking adjustment at x, prove sharpness.

3.4 Zolotarev and Wasserstein: extremisers and correction terms

Integration by parts in (1.2) gives (2.3); the boundary terms vanish by the finite first moment. Its kernel variation gives T, W_1, W_2. The polynomial-test theorem gives the corresponding higher-order assertion, including every required cancellation. For any nonnegative weight, exchanging the two suprema over tests and points proves that weighting an exact envelope gives the exact weighted factor, including when the result is infinite.

3.5 Fortet–Mourier refinements

For a prescribed value $h(x) = c$, the function in (1.15) is the largest admissible value to the left and the smallest to the right. The two kernel signs therefore prove the formula. For (1.16), the continuous measure is nonpositive and its negative mass is a. The extremiser is $h(t) = \max\{-B, B - L|t-x|\}$, giving the displayed truncated-cone integral. The upper bounds follow by intersecting the bounded and Lipschitz classes. Differentiating the equation once gives (1.14), so it does not need a first moment.

4 Why the missing bounds cannot hold

4.1 Polynomial freedom at low derivative orders

Whenever a polynomial of degree less than r has a nonzero solution derivative of order n, arbitrary multiples invalidate an H_r-only bound. Moment matching for a Zolotarev metric cancels the polynomial's contribution to the difference of expectations, not its contribution to an individual Stein solution. Anchoring lower test derivatives is a legitimate different class; it does not justify deleting correction terms for the unanchored class used here.

4.2 Local smooth counterexamples at high derivative orders

Fix a regular interior point x_0 and choose $h_\varepsilon(t) = \varepsilon^r \psi((t-x_0)/\varepsilon)$ with ψ smooth and compactly supported and $\|\psi^{(r)}\|_\infty \le 1$. Differentiating the equation $r+1$ times exhibits the leading term $a(x_0)h_\varepsilon^{(r+1)}(x_0)$ in $f^{(r+2)}(x_0)$. All other terms have smaller order in ε^{-1}. Choosing a nonzero jet at x_0 proves divergence. The same construction with the relevant higher jet works for every $n \ge r+2$. The case $r=0$ proves the bounded-test obstruction. For $r=1$ these bumps also belong to a fixed rectangular Fortet–Mourier class for sufficiently small ε. A weight positive at x_0 does not remove the failure.

4.3 Boundary divergence is a different obstruction

A finite pointwise envelope with infinite spatial supremum can often be repaired by a weight. This is distinct from failure at a fixed interior point, from a missing target moment, and from absence of a classical derivative at a nonsmooth point of the density. The tables distinguish these four issues; their entries must not be interchanged.

5 Evaluating and using the bounds

5.1 Numerical shape constants and explicit reference values

Use the reciprocal-Gamma identities

$$F(x) = \frac{\Gamma(\alpha, b/x)}{\Gamma(\alpha)}, \qquad Q(x) = \frac{\gamma(\alpha, b/x)}{\Gamma(\alpha)}, \qquad \mathbb{E}Z^r = b^r \frac{\Gamma(\alpha - r)}{\Gamma(\alpha)} \quad (r < \alpha).$$

Here $\Gamma(s,t)$ and $\gamma(s,t)$ are the upper and lower incomplete Gamma functions. The endpoint values $1/b$ and $2/b$ are particularly useful checks on numerical envelopes; they need not be the global maxima. For example, at $\alpha = 3, b = 1$, the exact second-derivative Wasserstein envelope is $2/x^2$, with infinite ordinary factor and weighted factor 2. At $\alpha = 3/2$ use W_2 instead: its x^2-weighted tail limit is $4/\alpha = 8/3$, so a uniform factor 2 would be false.

Scale. For $Y = bZ_1$, where $Z_1 \sim \mathrm{InvGamma}(\alpha, 1)$, $g_H^{(n)}(bx) = b^{-n-1} f_{\widetilde{H}}^{(n)}(x)$. An H_r factor acquires b^{r-n-1} ordinarily and b^{r-n+1} with the weight y^2. For indicator and bounded tests set $r = 0$.

5.2 Stable evaluation and scaling

Compute the smaller tail directly rather than subtracting it from one. Evaluate positive truncated moments by quadrature or special functions; subtractive recurrences are best used away from extreme tails. Use logarithms for small densities. Include analytic endpoint limits and exceptional-point asymptotics when maximising an envelope. A grid or a local optimiser alone does not certify a global maximum. The exact supremum definitions remain valid independently of numerical optimisation; rounded numerical values, where displayed, are illustrations.

5.3 How the weights enter approximation arguments

Write the equation as $wf' + \gamma f = h - \mathbb{E}h(Z)$. Suppose a comparison variable satisfies $\mathbb{E}[-\gamma(X)f(X)] = \mathbb{E}[T_X f'(X)] + \mathbb{E}[R_X f(X)]$ for the solutions under consideration, with all expectations finite. Then

$$\mathbb{E}h(X) - \mathbb{E}h(Z) = \mathbb{E}[(w(X) - T_X) f_h'(X)] - \mathbb{E}[R_X f_h(X)].$$

For a test class with envelopes E_0, E_1, this gives

$$d_{\mathcal{H}}(X, Z) \le \mathbb{E}[|w(X) - T_X|E_1(X)] + \mathbb{E}[|R_X|E_0(X)].$$

One may replace $E_1(X)$ by an ordinary factor or by $\mathfrak{C}_\omega[E_1]/\omega(X)$ where $\omega(X) > 0$. The pointwise version often retains information lost by either uniform factor. At exceptional points use the limiting equation only when it is defined for the chosen tests and when the assumed comparison identity includes that convention.

Relation to the literature and scope. The kernel construction follows the framework of Rapin and Swan [1]; the proofs above make the formulas used here self-contained. The iterative approach of Döbler, Gaunt and Vollmer [2] provides a complementary way to retain lower derivative controls. Exact envelope formulas, their one-dimensional supremum definitions, and convenient upper bounds are distinguished throughout. No blanket claim of numerical improvement over every previously published factor is made.

References

[1] F. Rapin and Y. Swan (2026). *Representations of solutions to Stein equations and their derivatives*. arXiv:2609.17881.

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[3] G. Afendras and N. Papadatos (2015). Integrated Pearson family and orthogonality of the Rodrigues polynomials: a review including new results and an alternative classification of the Pearson system. *Applicationes Mathematicae* **42**, 231–267. arXiv:1205.2903.