Gamma Stein envelopes and factors: weighted and unweighted bounds by test class

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Abstract

We organise pointwise envelopes and uniform factors for the Gamma Stein equation by test class: Kolmogorov, bounded measurable, Wasserstein, Fortet–Mourier, and integer-order Zolotarev. The first section gives the bounds at derivative orders zero, one, two, and general order, with separate ordinary and weighted factors. A kernel array built from integrated Gamma tails yields three neighbouring derivative representations. Under a bound on the rth test derivative, the solution derivatives of orders $r-1$ and r admit finite unweighted factors, whereas order $r+1$ has the sharp envelope $2/(bx)$ and the sharp x-weighted factor $2/b$. Explicit endpoint asymptotics classify the admissible power weights. Kolmogorov tests admit exact piecewise envelopes at every order, but require weights adapted to both endpoints beyond order one. We also give elementary bounds on the shape constants, exact Fortet–Mourier envelopes at orders zero and one, and counterexamples for unavailable bounds. All results are on the positive half-line and hold for every positive shape and scale.

Keywords: Gamma approximation; Stein equation; Stein factors; weighted derivative bounds; kernel representation; probability metrics.

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1 Results at a glance and by test class

1.1 Equation, normalisation, and the meaning of a weighted factor

Fix a shape $a > 0$ and a *scale* $b > 0$. Throughout, $G \sim \Gamma(a, b)$ has density

$$p(x) = \frac{x^{a-1}e^{-x/b}}{\Gamma(a)b^a}, \qquad x > 0, \qquad \mathbb{E}G = ab, \quad \mathrm{Var}(G) = ab^2. \tag{1.1}$$

Write $P(x) = \Pr(G \le x)$ and $\overline{P}(x) = 1 - P(x)$. We use the same operator normalisation as Bailly, Rapin, Swan and von Sachs [1]:

$$bxf_h'(x) + (ab - x)f_h(x) = h(x) - \mathbb{E}h(G), \qquad x > 0. \tag{1.2}$$

Its canonical solution is

$$f_h(x) = \frac{1}{bxp(x)} \int_0^x \{h(v) - \mathbb{E}h(G)\}p(v)\,dv = -\frac{1}{bxp(x)} \int_x^\infty \{h(v) - \mathbb{E}h(G)\}p(v)\,dv. \tag{1.3}$$

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It is defined for $\mathbb{E}|h(G)| < \infty$ and satisfies the Stein equation almost everywhere, and everywhere for continuous h. Adding a constant to h leaves f_h unchanged. Under the smoothness assumptions below, this is the unique solution of at most polynomial growth at infinity.

All norms and derivatives in this paper are on $(0,\infty)$. For bounded tests, $\|h\|_\infty = \sup_{x>0}|h(x)|$ and $\operatorname{osc}(h) = \sup h - \inf h$. Derivative norms are essential suprema. For $r \geq 1$, the notation $\|h^{(r)}\|_\infty < \infty$ includes the assumptions that $h \in C^{r-1}(0,\infty)$ and $h^{(r-1)}$ is locally absolutely continuous with essentially bounded derivative. Thus $h^{(r-1)}$ is globally Lipschitz. These assumptions imply polynomial growth at infinity and finite one-sided limits of lower derivatives at zero.

An *envelope* $E_n(x)$ bounds $|f_h^{(n)}(x)|$ over the indicated normalised class. The ordinary factor is $\sup_{x>0} E_n(x)$; the x-weighted factor is $\sup_{x>0} xE_n(x)$. In particular,

$$\|xf_h^{(n)}\|_\infty = \operatorname*{ess\,sup}_{x>0} x|f_h^{(n)}(x)|$$

is the norm of the derivative multiplied by x, *not* the norm of $(xf_h)^{(n)}$. An exact envelope is the pointwise class supremum, with limiting extremisers allowed. Highest derivatives hold almost everywhere unless more smoothness is assumed; Kolmogorov derivatives of positive order are taken away from the test's jump. For the Stein-kernel weight $\tau(x) = bx$, simply multiply every x-weighted factor by b; for example, the factor $2/b$ for $xf_h^{(r+1)}$ becomes 2 for $\tau f_h^{(r+1)}$.

The quantities needed in the tables. Set $P_0 = \overline{P}_0 = p$ and, for $j \geq 1$, define

$$P_j(x) = \int_0^x P_{j-1}(v)\,dv, \qquad \overline{P}_j(x) = \int_x^\infty \overline{P}_{j-1}(v)\,dv. \tag{1.4}$$

Thus $P_1 = P$, $\overline{P}_1 = \overline{P}$. Introduce

$$\mathcal{A}_{a,b}(x) = \frac{P(x)\overline{P}(x)}{bxp(x)}, \tag{1.5}$$

$$\mathcal{D}_{a,b,n}(x) = \frac{2n!P_{n+1}(x)\overline{P}_{n+1}(x)}{b^{n+1}x^{n+1}p(x)}, \qquad n \geq 1. \tag{1.6}$$

The exact shape constants are the following one-dimensional suprema:

$$u_a = \sup_{y>0} \mathcal{A}_{a,1}(y), \qquad v_a = \sup_{y>0} y\mathcal{A}_{a,1}(y), \tag{1.7}$$

$$c_{a,n} = \sup_{y>0} \mathcal{D}_{a,1,n}(y), \qquad d_{a,n} = \sup_{y>0} y\mathcal{D}_{a,1,n}(y). \tag{1.8}$$

All are finite. They give the factors at arbitrary scale because

$$\mathcal{A}_{a,b}(x) = b^{-1}\mathcal{A}_{a,1}(x/b), \qquad \mathcal{D}_{a,b,n}(x) = b^{-1}\mathcal{D}_{a,1,n}(x/b). \tag{1.9}$$

If an easily evaluated upper bound is preferred to optimisation, then

$$c_{a,n} \leq \frac{2}{n}, \qquad d_{a,n} \leq 2\left(1 + \frac{(a-1)_+}{n}\right), \qquad n \geq 1. \tag{1.10}$$

For u_a, v_a, explicit finite bracketing bounds are given in section 5. Those bounds are conservative; the exact envelopes and the constants (1.7)–(1.8) retain the shape dependence.

The results are organised to separate three issues: existence of a derivative, a bound at an interior point, and a finite uniform factor. All shapes $a > 0$ are allowed, including the cases where the density is unbounded at zero. The low-order kernel bounds in [1] motivate this treatment. With the present notation their constants c_a, d_a, e_a are $c_{a,1}, d_{a,1}, c_{a,2}$. The arguments below give the all-order formulas and their endpoint and weight classifications directly.

Normalised class	$\|f_h\|_\infty$	$\|f_h'\|_\infty$	$\|f_h''\|_\infty$
Kolmogorov	u_a/b	∞	∞ (piecewise)
Bounded, $\operatorname{osc}(h) \le 1$	u_a/b	∞	none
Wasserstein, $\operatorname{Lip}(h) \le 1$	1	$c_{a,1}/b$	∞
Fortet–Mourier, $\|h\|_\infty, \operatorname{Lip}(h) \le 1$	$\le \min(2u_a/b, 1)$	$\le c_{a,1}/b$	∞
Zolotarev $r = 2$, $\|h''\|_\infty \le 1$	none	$1/2$	$c_{a,2}/b$

Table 1: Ordinary factors at orders zero, one, and two. A finite entry without $\le$ is optimal. The symbol ∞ indicates an exact finite pointwise envelope with infinite spatial supremum, except for the Fortet–Mourier entry, whose class supremum is also infinite. "None" means that no finite class envelope exists even at a fixed interior point.

Normalised class	$\|xf_h\|_\infty$	$\|xf_h'\|_\infty$	$\|xf_h''\|_\infty$
Kolmogorov	v_a	$1/b$	∞ (piecewise)
Bounded, $\operatorname{osc}(h) \le 1$	v_a	$1/b$	none
Wasserstein, $\operatorname{Lip}(h) \le 1$	∞	$d_{a,1}$	$2/b$
Fortet–Mourier, $\|h\|_\infty, \operatorname{Lip}(h) \le 1$	$\le 2v_a$	$\le \min(2/b, d_{a,1})$	$2/b$
Zolotarev $r = 2$, $\|h''\|_\infty \le 1$	none	∞	$d_{a,2}$

Table 2: The corresponding x-weighted factors. Weighting repairs the bounded-test first derivative and the Wasserstein second derivative. It need not improve other entries: for example, $f_h = -1$ for $h(x) = x$, so $\|xf_h\|_\infty = \infty$.

1.2 Kolmogorov tests: piecewise derivatives and the required weights

For $z > 0$, take $h_z(v) = \mathbf{1}_{\{v \le z\}}$ and $f_z = f_{h_z}$. The associated metric is $d_{\mathrm{K}}(X, G) = \sup_z |\Pr(X \le z) - \Pr(G \le z)|$ for positive X. Define, for every integer $n \ge 0$,

$$\mathcal{Q}_{a,b,n}(x) = \frac{n!}{b^{n+1}x^{n+1}p(x)} \max\{\overline{P}(x)P_{n+1}(x),\ P(x)\overline{P}_{n+1}(x)\}. \tag{1.11}$$

Theorem 1.1 (Exact Kolmogorov envelopes)**.** *For $x \ne z$,*

$$f_z^{(n)}(x) = \frac{n!}{b^{n+1}x^{n+1}p(x)} \begin{cases} \overline{P}(z)P_{n+1}(x), & x < z, \\ (-1)^n P(z)\overline{P}_{n+1}(x), & x > z. \end{cases} \tag{1.12}$$

Consequently $\sup_{z \ne x} |f_z^{(n)}(x)| = \mathcal{Q}_{a,b,n}(x)$*. At order zero the formula extends continuously to $x = z$.*

Order	Exact envelope	Ordinary factor	x-weighted factor
0	$\mathcal{A}_{a,b}(x)$	u_a/b	v_a
1	$\mathcal{Q}_{a,b,1}(x)$	∞	$1/b$
2	$\dfrac{2\max\{\overline{P}P_3, P\overline{P}_3\}}{b^3x^3p}(x)$	∞	∞
$n \ge 2$	$\mathcal{Q}_{a,b,n}(x)$	∞	∞

For direct evaluation of the order-one entry,

$$\mathcal{Q}_{a,b,1}(x) = \frac{1}{bx} \max\{\overline{P}(x) + (x - ab)\mathcal{A}_{a,b}(x),\ P(x) - (x - ab)\mathcal{A}_{a,b}(x)\}. \tag{1.13}$$

The two terms inside the maximum are nonnegative and sum to one. The jump is $f'_z(z+) - f'_z(z-) = -1/(bz)$; distributional higher derivatives therefore contain singular terms.

For $n \geq 1$ the exact endpoint asymptotics are

$$\mathcal{Q}_{a,b,n}(x) \sim \frac{(a)_n}{ab}\, x^{-n} \quad (x \downarrow 0), \qquad \mathcal{Q}_{a,b,n}(x) \sim \frac{1}{b^n x} \quad (x \to \infty), \tag{1.14}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$. Thus, for $n \geq 2$, no power weight x^q yields a finite factor: zero requires $q \geq n$, whereas infinity requires $q \leq 1$. An explicit weight that does work at every positive order is

$$\omega_{n,b}(x) = \frac{x^n}{(b+x)^{n-1}}, \qquad k_{a,n} = \sup_{y>0} \frac{y^n}{(1+y)^{n-1}}\, \mathcal{Q}_{a,1,n}(y) < \infty. \tag{1.15}$$

It gives the sharp weighted factor

$$\sup_{z>0} \|\omega_{n,b} f_z^{(n)}\|_\infty = b^{-n} k_{a,n}, \qquad n \geq 1, \tag{1.16}$$

with piecewise derivatives. At $n = 1$, $\omega_{1,b} = x$ and $k_{a,1} = 1$. This distinction is essential: multiplying by x^n alone repairs the origin but fails at infinity for $n \geq 2$.

1.3 Bounded measurable tests

The class $\operatorname{osc}(h) \leq 1$ induces total variation in the convention $d_{\mathrm{TV}}(X, G) = \sup_B |\Pr(X \in B) - \Pr(G \in B)|$. The class $\|h\|_\infty \leq 1$ induces twice that distance.

Theorem 1.2 (Bounded-test envelopes and factors)**.** *Let $H_0 = \operatorname{osc}(h) < \infty$. The following finite bounds are sharp:*

Order	Pointwise bound	Ordinary factor	x-weighted factor
0	$\mathcal{A}_{a,b}(x) H_0$	u_a/b	v_a
1	$H_0/(bx)$ a.e.	∞	$1/b$
2	no finite class envelope	none	none
$n \geq 2$	no finite class envelope	none	none

Factors in the last two columns multiply H_0. If $\|h\|_\infty \leq B$, replace H_0 by $2B$; the resulting single-norm bounds are also sharp.

The higher-order negative statements hold even for smooth bounded tests. A weight which is positive near an interior point cannot restore these missing bounds. This differs from Kolmogorov tests, whose special piecewise form permits derivatives of all orders.

1.4 Wasserstein tests

Let $L = \operatorname{Lip}(h) = \|h'\|_\infty$. The unit class induces the Wasserstein distance when the compared random variables have finite first moments.

Theorem 1.3 (Wasserstein envelopes and factors)**.** *The solution belongs to $C^1(0, \infty)$, its first derivative is locally absolutely continuous, and*

Order	Exact envelope times L	Ordinary factor	x-weighted factor
0	L	1	∞
1	$\mathcal{D}_{a,b,1}(x) L$	$c_{a,1}/b$	$d_{a,1}$
2	$2L/(bx)$ a.e.	∞	$2/b$
$n \geq 3$	no finite class envelope	none	none

The finite envelopes and factors are exact; factors multiply L.

In terms of the density and distribution function alone, set

$$U(x) = (x - ab)P(x) + bxp(x) = P_2(x), \tag{1.17}$$
$$V(x) = (ab - x)\overline{P}(x) + bxp(x) = \overline{P}_2(x). \tag{1.18}$$

Then

$$\mathcal{D}_{a,b,1}(x) = \frac{2U(x)V(x)}{b^2x^2p(x)}, \quad \|f_h'\|_\infty \le \frac{2L}{b}, \quad \|xf_h'\|_\infty \le 2\max(1,a)L. \tag{1.19}$$

The last two are elementary upper bounds; $c_{a,1}/b$ and $d_{a,1}$ are the optimal factors. The order-zero extremiser is $h(v) = v$, for which $f_h = -1$. The order-one envelope is attained by $h(v) = -|v - x|$.

1.5 Fortet–Mourier tests: size and Lipschitz constraints together

First use the rectangular convention $\mathcal{H}_{B,L} = \{h : \|h\|_\infty \le B,\ \mathrm{Lip}(h) \le L\}$ on $(0,\infty)$. The usual rectangular unit class has $B = L = 1$.

Proposition 1.4 (Fortet–Mourier bounds)**.** *For $B, L > 0$ and $h \in \mathcal{H}_{B,L}$,*

Order	Pointwise upper bound	Ordinary upper factor	x-weighted upper factor
0	$\min\{2B\mathcal{A}_{a,b}(x), L\}$	$\min\{2Bu_a/b, L\}$	$2Bv_a$
1	$\min\{2B/(bx), L\mathcal{D}_{a,b,1}(x)\}$	$Lc_{a,1}/b$	$\min\{2B/b, Ld_{a,1}\}$
2	$2L/(bx)$ a.e.	∞	$2L/b$
$n \ge 3$	no finite class envelope	none	none

The displayed factors at orders zero and one are upper bounds, not claims of optimality under both constraints. At order two the weighted factor $2L/b$ is optimal for every $B > 0$, and the unweighted class supremum is infinite. If $B = 0$ or $L = 0$, the solutions vanish.

Exact refinements at orders zero and one. Let $S = P/(bxp)$ and $R = \overline{P}/(bxp)$. For $\alpha \in [-B, B]$, put $u_{\alpha,x}(v) = \mathrm{clip}(\alpha + L(x - v), -B, B)$, where $\mathrm{clip}(t, -B, B) = \max(-B, \min(t, B))$. Define

$$F_0(B, L; x) = \max_{\alpha\in[-B,B]} \left\{ R(x) \int_0^x u_{\alpha,x}(v)p(v)\,dv - S(x) \int_x^\infty u_{\alpha,x}(v)p(v)\,dv \right\}, \tag{1.20}$$
$$q_x(v) = bx\{-R'(x)\mathbf{1}_{\{v\le x\}} + S'(x)\mathbf{1}_{\{v>x\}}\}p(v), \tag{1.21}$$
$$F_1(B, L; x) = \frac{1}{bx} \int_0^\infty \min\{L|v - x|, 2B\}\, q_x(v)\,dv. \tag{1.22}$$

Then q_x is a probability density and

$$\sup_{h\in\mathcal{H}_{B,L}} |f_h(x)| = F_0(B, L; x), \qquad \sup_{h\in\mathcal{H}_{B,L}} |f_h'(x)| = F_1(B, L; x). \tag{1.23}$$

Taking $\sup_x F_j$ or $\sup_x xF_j$ gives the exact ordinary or weighted factor. Thus the first two rows admit explicit optimisation formulas when the simpler bounds in proposition 1.4 are insufficient.

The sum-norm convention. If the unit class is instead $\|h\|_\infty + \mathrm{Lip}(h) \le 1$, valid envelopes are

$$|f_h(x)| \le \frac{2\mathcal{A}_{a,b}(x)}{1+2\mathcal{A}_{a,b}(x)}, \qquad |f_h'(x)| \le \frac{2\mathcal{D}_{a,b,1}(x)}{2+bx\mathcal{D}_{a,b,1}(x)}, \qquad |f_h''(x)| \le \frac{2}{bx}. \tag{1.24}$$

Consequently one may use

$$\|f_h\|_\infty \le \frac{2u_a}{b+2u_a}, \qquad \|xf_h\|_\infty \le 2v_a, \tag{1.25}$$

$$\|f_h'\|_\infty \le c_{a,1}/b, \qquad \|xf_h'\|_\infty \le \frac{2d_{a,1}}{2+bd_{a,1}}, \qquad \|xf_h''\|_\infty \le 2/b. \tag{1.26}$$

The last constant is sharp; the other displayed constants are upper bounds. Exact order-zero and order-one envelopes are $\sup_{0\le B\le 1} F_j(B, 1-B; x)$, $j = 0, 1$, with zero endpoint cases. At order two the unweighted supremum is infinite; at orders $n \ge 3$ there is no finite pointwise envelope. Both Fortet–Mourier conventions are coordinate-dependent, so scaling changes the Lipschitz constraint.

1.6 Integer-order Zolotarev tests

For an integer $r \ge 1$, let

$$\mathcal{Z}_r = \{h : h^{(r-1)} \text{ is globally Lipschitz on } (0,\infty),\ \|h^{(r)}\|_\infty \le 1\},$$

with the preceding regularity convention. Lower derivatives are not bounded or anchored. The ideal metric is $\zeta_r(X,G) = \sup_{h\in\mathcal{Z}_r} |\mathbb{E}h(X) - \mathbb{E}h(G)|$. For positive X with finite rth moment, it is finite precisely when the moments of orders $1,\dots,r-1$ match those of G. For $r = 1$ the condition is empty and ζ_1 is Wasserstein. In particular, ζ_2 needs equal means; equal variances are not required for its finiteness. Fractional-order classes are outside our scope.

Theorem 1.5 (All-order Zolotarev classification)**.** *For $h \in \mathcal{Z}_r$, the solution belongs to $C^r(0,\infty)$, with locally absolutely continuous rth derivative. Its envelopes and factors are*

Solution order	Exact envelope	Ordinary factor	x-weighted factor
$0 \le n \le r-2$	no finite class envelope	none	none
$r-1$	$1/r$	$1/r$	∞
r	$\mathcal{D}_{a,b,r}(x)$	$c_{a,r}/b$	$d_{a,r}$
$r+1$	$2/(bx)$ a.e.	∞	$2/b$
$n \ge r+2$	no finite class envelope	none	none

The first row is empty for $r = 1$. All finite entries are sharp. For a general $H_r = \|h^{(r)}\|_\infty$, multiply the three finite envelopes and their finite factors by H_r.

The requested low orders can be read off without reindexing:

Class	Order 0	Order 1	Order 2
$\mathcal{Z}_1$	1; factors $(1,\infty)$	$\mathcal{D}_{a,b,1}$; $(c_{a,1}/b, d_{a,1})$	$2/(bx)$; $(\infty, 2/b)$
$\mathcal{Z}_2$	none	$1/2$; $(1/2,\infty)$	$\mathcal{D}_{a,b,2}$; $(c_{a,2}/b, d_{a,2})$
$\mathcal{Z}_3$	none	none	$1/3$; $(1/3,\infty)$
$\mathcal{Z}_r$, $r \ge 4$	none	none	none

Each pair lists the ordinary and x-weighted factors, in that order. For an explicit second-order envelope, let

$$\begin{aligned} I(x) &= \frac{(x-b(a+1))U(x)+bxP(x)}{2} = P_3(x), \\ J(x) &= \frac{(b(a+1)-x)V(x)+bx\overline{P}(x)}{2} = \overline{P}_3(x). \end{aligned} \tag{1.27}$$

Then

$$\mathcal{D}_{a,b,2}(x) = \frac{4I(x)J(x)}{b^3x^3p(x)}. \tag{1.28}$$

An uncontrolled polynomial of degree below r can make individual Stein solutions arbitrarily large; moment matching cancels its contribution to the metric, not its contribution to every derivative norm.

1.7 Combining smoothness controls and choosing other weights

Put $H_0 = \operatorname{osc}(h)$ and $H_j = \|h^{(j)}\|_\infty$ for $j \geq 1$. Each term below is available under its own regularity assumption; omit terms for which the indicated control is unavailable:

$$|f_h(x)| \leq \min\{\mathcal{A}_{a,b}(x)H_0, H_1\}, \tag{1.29}$$

$$|f_h'(x)| \leq \min\{H_0/(bx), \mathcal{D}_{a,b,1}(x)H_1, H_2/2\}, \tag{1.30}$$

$$|f_h''(x)| \leq \min\{2H_1/(bx), \mathcal{D}_{a,b,2}(x)H_2, H_3/3\}, \tag{1.31}$$

$$|f_h^{(n)}(x)| \leq \min\{2H_{n-1}/(bx), \mathcal{D}_{a,b,n}(x)H_n, H_{n+1}/(n+1)\}, \quad n \geq 2. \tag{1.32}$$

Multiply an envelope by the desired nonnegative weight before taking its supremum. For instance,

$$\|xf_h^{(n)}\|_\infty \leq \min\{2H_{n-1}/b, d_{a,n}H_n\}, \quad n \geq 2, \tag{1.33}$$

when both controls are available. The term involving H_{n+1} alone has infinite x-weighted factor, so it supplies no additional uniform term here. Each single-seminorm envelope is sharp; sharpness of their minimum under simultaneous constraints is not asserted.

For reference, the complete power-weight classification for these exact envelopes is

Envelope being multiplied by x^q	Exponents giving a finite factor
$\mathcal{A}_{a,b}$ (bounded tests or Kolmogorov order 0)	$0 \leq q \leq 1$
$1/r$ (Zolotarev solution order $r-1$)	$q = 0$
$\mathcal{D}_{a,b,r}$ (Zolotarev solution order r)	$0 \leq q \leq 1$
$1/(bx)$ or $2/(bx)$	$q = 1$
$\mathcal{Q}_{a,b,n}$, $n \geq 1$ (Kolmogorov)	$n \leq q \leq 1$

Thus the last row is empty for $n \geq 2$. For $0 \leq q \leq 1$, exact intermediate diagonal factors and a convenient upper bound are

$$\|x^q f_h^{(r)}\|_\infty \leq b^{q-1}c_{a,r}^{[q]}H_r, \qquad c_{a,r}^{[q]} := \sup_{y>0} y^q\mathcal{D}_{a,1,r}(y) \leq c_{a,r}^{1-q}d_{a,r}^q. \tag{1.34}$$

The analogous bounded-test constant is $b^{q-1}\sup_y y^q\mathcal{A}_{a,1}(y) \leq b^{q-1}u_a^{1-q}v_a^q$. These formulas make the weight and the scale dependence explicit.

2 The Gamma kernel calculus

2.1 Integrated tails and weighted Mills ratios

For $j \geq 1$, Fubini's theorem gives

$$P_j(x) = \frac{\mathbb{E}[(x-G)_+^{j-1}]}{(j-1)!}, \qquad \overline{P}_j(x) = \frac{\mathbb{E}[(G-x)_+^{j-1}]}{(j-1)!}, \tag{2.1}$$

where a zeroth power denotes the corresponding indicator. The identity $(bxp)' = (ab-x)p$ implies the recurrences

$$(j+1)P_{j+2}(x) = (x - b(a+j))P_{j+1}(x) + bxP_j(x), \tag{2.2}$$

$$(j+1)\overline{P}_{j+2}(x) = (b(a+j) - x)\overline{P}_{j+1}(x) + bx\overline{P}_j(x), \quad j \geq 0. \tag{2.3}$$

They follow by integration by parts in the truncated moments and include (1.17)–(1.18) and (1.27) as special cases.

The weighted Mills ratios $S = P/(bxp)$ and $R = \overline{P}/(bxp)$ have positive integral representations

$$S^{(n)}(x) = b^{-n-1}\int_0^1 (1-t)^n t^{a-1} e^{(x/b)(1-t)}\,dt = \frac{n!P_{n+1}(x)}{b^{n+1}x^{n+1}p(x)}, \tag{2.4}$$

$$(-1)^n R^{(n)}(x) = b^{-n-1}\int_1^\infty (t-1)^n t^{a-1} e^{-(x/b)(t-1)}\,dt = \frac{n!\overline{P}_{n+1}(x)}{b^{n+1}x^{n+1}p(x)}. \tag{2.5}$$

The substitution $v = xt$ proves the $n = 0$ identities; differentiation under the integral sign proves the general formulas. Thus S is absolutely monotone and R is completely monotone on $(0,\infty)$. No log-concavity restriction on p is needed.

2.2 The array, its signs, and its three special masses

For integers $i, j \geq 0$, define

$$K^{i,j}(x,v) = \frac{1}{p(v)}\left\{(-1)^j R^{(i)}(x)P_j(v)\mathbf{1}_{\{v\leq x\}} - S^{(i)}(x)\overline{P}_j(v)\mathbf{1}_{\{v>x\}}\right\}. \tag{2.6}$$

Here x is the evaluation point and v is the integration variable. The first index records the solution derivative, the second the test derivative. This ordering of the arguments is the reverse of that used for the low-order kernels in [1].

Lemma 2.1 (Mass identities)**.** *For $i, j \geq 0$,*

$$\mathbb{E}K^{i,j}(x,G) = \frac{i!\{(-1)^{i+j}\overline{P}_{i+1}(x)P_{j+1}(x) - P_{i+1}(x)\overline{P}_{j+1}(x)\}}{b^{i+1}x^{i+1}p(x)}, \tag{2.7}$$

$$\mathbb{E}|K^{i,j}(x,G)| = \frac{i!\{\overline{P}_{i+1}(x)P_{j+1}(x) + P_{i+1}(x)\overline{P}_{j+1}(x)\}}{b^{i+1}x^{i+1}p(x)}. \tag{2.8}$$

When $i+j$ is even the kernel is positive left of x and negative right of x; when $i+j$ is odd it is negative on both sides. In particular,

$$\mathbb{E}K^{n,n}(x,G) = 0, \qquad \mathbb{E}|K^{n,n}(x,G)| = \mathcal{D}_{a,b,n}(x), \quad n \geq 1, \tag{2.9}$$

$$\mathbb{E}K^{n,n-1}(x,G) = -\frac{1}{bx}, \qquad \mathbb{E}|K^{n,n-1}(x,G)| = \frac{1}{bx}, \quad n \geq 1, \tag{2.10}$$

$$\mathbb{E}K^{n,n+1}(x,G) = -\frac{1}{n+1}, \qquad \mathbb{E}|K^{n,n+1}(x,G)| = \frac{1}{n+1}, \quad n \geq 0. \tag{2.11}$$

At $n = 0$ the diagonal is centred and its absolute mass is $2\mathcal{A}_{a,b}(x)$.

Proof. Insert (2.4)–(2.5) into (2.6) and integrate the two disjoint half-lines. This proves (2.7)–(2.8) and the sign claims. The recurrences imply

$$n!\{P_{n+1}(x)\overline{P}_n(x) + \overline{P}_{n+1}(x)P_n(x)\} = (bx)^n p(x), \qquad n \geq 0. \tag{2.12}$$

For $n = 0$ this says $(P + \overline{P})p = p$. For $n \geq 1$, substituting the recurrences at index $n-1$ cancels the terms containing $x - b(a+n-1)$ and reduces the identity to the preceding index. Dividing (2.12) by $b^{n+1}x^{n+1}p(x)$ gives (2.10); using the identity at index $n+1$ gives (2.11). The diagonal formulas are immediate. □

2.3 Representations at every derivative order

Theorem 2.2 (Three neighbouring representations)**.** *For bounded measurable h,*

$$f_h(x) = \mathbb{E}[K^{0,0}(x,G)h(G)], \tag{2.13}$$

$$f_h'(x) = \mathbb{E}[K^{1,0}(x,G)\{h(G) - h(x)\}] \quad a.e. \tag{2.14}$$

For $r \geq 1$ and $\|h^{(r)}\|_\infty < \infty$ under our regularity convention,

$$f_h^{(r-1)}(x) = \mathbb{E}[K^{r-1,r}(x,G)h^{(r)}(G)], \tag{2.15}$$

$$f_h^{(r)}(x) = \mathbb{E}[K^{r,r}(x,G)h^{(r)}(G)], \tag{2.16}$$

$$f_h^{(r+1)}(x) = \mathbb{E}[K^{r+1,r}(x,G)\{h^{(r)}(G) - h^{(r)}(x)\}]. \tag{2.17}$$

The first two hold at every $x > 0$. The last holds almost everywhere, in particular at Lebesgue points of $h^{(r)}$.

Proof. Splitting the integral in (1.3) yields (2.13). Differentiation supplies the boundary term $h(x)/(bx)$. Its cancellation with $\mathbb{E}K^{1,0} = -1/(bx)$ yields (2.14).

For smooth tests of polynomial growth, integration by parts gives

$$\mathbb{E}[K^{i,j}(x,G)h^{(j)}(G)] = \mathbb{E}[K^{i,j+1}(x,G)h^{(j+1)}(G)] + \mathbb{E}[K^{i,j}(x,G)]h^{(j)}(x). \tag{2.18}$$

The terms at zero vanish because $P_{j+1}(v) = O(v^{a+j})$; those at infinity vanish by exponential tail decay. The diagonal has mean zero, so (2.18) moves a diagonal representation one test derivative to the right without a correction term. Differentiating the diagonal instead gives

$$f_h^{(i+1)}(x) = \frac{h^{(i)}(x)}{bx} + \mathbb{E}[K^{i+1,i}(x,G)h^{(i)}(G)]. \tag{2.19}$$

The boundary coefficient is $1/(bx)$ by (2.12). Applying (2.18) to the remaining kernel term cancels the explicit boundary contribution, producing the next diagonal. Induction proves the three representations.

For the stated regularity, extend h below zero by its Taylor polynomial of degree $r-1$ at zero. The extension has rth weak derivative bounded by $\|h^{(r)}\|_\infty$. Mollification preserves this bound and supplies common polynomial majorants for lower derivatives. Dominated convergence on the kernel integrals gives the first two identities; the differentiated Stein equation gives the last one almost everywhere. Local regularity of (1.2) on $(0,\infty)$ gives $f_h \in C^r$ and local absolute continuity of $f_h^{(r)}$. □

More generally, for $\|h^{(k)}\|_\infty < \infty$ and $0 \leq n \leq j \leq k$, iteration of (2.18) gives

$$f_h^{(n)}(x) = \mathbb{E}[K^{n,j}(x,G)h^{(j)}(G)] + \sum_{\ell=n}^{j-1} \mathbb{E}[K^{n,\ell}(x,G)]h^{(\ell)}(x). \tag{2.20}$$

The sum is empty for $j = n$ and zero for $j = n+1$; farther away it must generally be retained. For example, $h(v) = v$ has $h'' = 0$ but $f_h = -1$.

3 Proofs of the envelopes, weights, and sharpness

3.1 Endpoint asymptotics and finiteness of the shape constants

From the truncated moments and a beta integral, as $x \downarrow 0$,

$$P_{n+1}(x) \sim \frac{x^{a+n}}{b^a \Gamma(a+n+1)}, \qquad \overline{P}_{n+1}(x) \longrightarrow \frac{b^n (a)_n}{n!}. \tag{3.1}$$

At infinity, for fixed n,

$$P_{n+1}(x) \sim \frac{x^n}{n!}, \qquad \overline{P}_{n+1}(x) \sim b^{n+1} p(x). \tag{3.2}$$

For the second relation, substitute $v = x + t$ in the upper truncated moment and use $p(x+t)/p(x) = (1 + t/x)^{a-1} e^{-t/b}$ with dominated convergence. It follows that

$$\mathcal{A}_{a,b}(x) \longrightarrow \frac{1}{ab} \quad (x \downarrow 0), \qquad \mathcal{A}_{a,b}(x) \sim \frac{1}{x} \quad (x \to \infty), \tag{3.3}$$

$$\mathcal{D}_{a,b,n}(x) \longrightarrow \frac{2}{b(a+n)} \quad (x \downarrow 0), \qquad \mathcal{D}_{a,b,n}(x) \sim \frac{2}{x} \quad (x \to \infty). \tag{3.4}$$

Together with continuity on the interior, these limits prove finiteness of all constants in (1.7)–(1.8) and the stated power-weight classifications. They also give the useful lower bounds

$$u_a \geq 1/a, \quad v_a \geq 1, \quad c_{a,n} \geq 2/(a+n), \quad d_{a,n} \geq 2. \tag{3.5}$$

Applying the same limits to (1.11) gives (1.14); the two endpoint limits of the expression defining $k_{a,n}$ are $(a)_n/a$ and 1. This proves its finiteness.

3.2 Elementary upper bounds for diagonal factors

Proof of (1.10). The lower Mills integral gives $S^{(n)} \leq b^{-1} S^{(n-1)}$. Since $\mathcal{D}_{a,b,n} = 2S^{(n)}\overline{P}_{n+1}$ and the absolute mass of $K^{n-1,n}$ is $1/n$,

$$\mathcal{D}_{a,b,n}(x) \leq \frac{2}{b} S^{(n-1)}(x) \overline{P}_{n+1}(x) \leq \frac{2}{bn}.$$

Also (2.10) gives $S^{(n)}(x)\overline{P}_n(x) \leq 1/(bx)$, hence

$$x\mathcal{D}_{a,b,n}(x) \leq \frac{2}{b} \frac{\overline{P}_{n+1}(x)}{\overline{P}_n(x)}.$$

To bound the ratio, integrate the derivative of $t^n p(x+t)$ over $t > 0$. The boundary terms vanish and

$$\frac{1}{b} \int_0^\infty t^n p(x+t)\, dt = n \int_0^\infty t^{n-1} p(x+t)\, dt + (a-1) \int_0^\infty \frac{t^n}{x+t} p(x+t)\, dt.$$

Using $0 \leq t/(x+t) \leq 1$ and the factorials in (2.1) yields $\overline{P}_{n+1}/\overline{P}_n \leq b(1 + (a-1)_+/n)$. This proves the second assertion. □

3.3 Kolmogorov and bounded-test envelopes

Proof of theorem 1.1. Formula (1.3) gives $f_z(x) = \overline{P}(z)S(x)$ for $x \leq z$ and $f_z(x) = P(z)R(x)$ for $x > z$. Differentiate on each side using (2.4)–(2.5). At fixed x, the two branch suprema are approached by $z \to x$ from the appropriate side, proving exactness of (1.11). For order one the two nonnegative terms in (1.13) sum to one, so $x\mathcal{Q}_{a,b,1} \leq 1/b$. The small-$x$ asymptotic shows equality of the supremum. All remaining factor statements follow from the endpoint limits and pointwise sharpness. □

Proof of the finite bounds in theorem 1.2. The kernel $K^{0,0}$ is centred with absolute mass $2\mathcal{A}_{a,b}$. Subtract the midpoint of the range of h to obtain $|f_h| \le \mathcal{A}_{a,b}\,\mathrm{osc}(h)$. The kernel $K^{1,0}$ is negative with absolute mass $1/(bx)$; (2.14) gives the first-derivative bound. The test $h_x = \mathbf{1}_{(0,x]}$ attains the order-zero envelope. For fixed $x > 0$ and $0 < \varepsilon < x$, the test $h_{x,\varepsilon} = \mathbf{1}_{[x-\varepsilon,x+\varepsilon]}$ has oscillation one, is constant near x, and satisfies

$$f'_{h_{x,\varepsilon}}(x) = \frac{1}{bx} + \mathbb{E}[K^{1,0}(x,G)h_{x,\varepsilon}(G)] \longrightarrow \frac{1}{bx}.$$

Smooth plateaux yield the same limit. The derivative is continuous near the evaluation point, so the construction also proves sharpness in essential-supremum norms. Rescaling $[0,1]$ to $[-B,B]$ gives the supremum-norm formulation. The impossibility at higher orders is proved in section 4. □

3.4 Zolotarev and Wasserstein: extremisers in all three bands

Proof of the finite entries in theorem 1.5. Take absolute values in theorem 2.2 and apply lemma 2.1. This gives H_r/r, $\mathcal{D}_{a,b,r}(x)H_r$, and $2H_r/(bx)$. The endpoint limits and scaling give their factors.

For sharpness at order $r-1$, take $h(v) = v^r/r!$. Its rth derivative is one and (2.15) gives $f_h^{(r-1)} = -1/r$. For order r, prescribe $h^{(r)}(v) = \mathbf{1}_{\{v<x\}} - \mathbf{1}_{\{v>x\}}$ and take successive antiderivatives on $(0,\infty)$. This matches the sign of $K^{r,r}$, so equality holds in the diagonal envelope. For order $r+1$, prescribe

$$h_{x,\varepsilon}^{(r)}(v) = \mathbf{1}_{\{|v-x|\le\varepsilon\}} - \mathbf{1}_{\{|v-x|>\varepsilon\}}, \qquad 0 < \varepsilon < x.$$

Then

$$f_{h_{x,\varepsilon}}^{(r+1)}(x) = -2\mathbb{E}[K^{r+1,r}(x,G)\mathbf{1}_{\{|G-x|>\varepsilon\}}] \longrightarrow \frac{2}{bx}.$$

These tests are smooth near x and can be mollified without increasing the controlled norm. Thus sharpness applies to ordinary pointwise derivatives where claimed and to essential-supremum factors. Taking suprema over x proves the optimality of the weighted constants and the infinite unweighted factor at order $r+1$. □

theorem 1.3 is the case $r = 1$. The master inequalities follow by combining available controls. For moment finiteness of ζ_r, subtract the Taylor polynomial of degree $r-1$ at zero. Its remainder is bounded by $x^r/r!$ for $h \in \mathcal{Z}_r$. Matching lower moments eliminates the polynomial, proving finiteness. A mismatched lower moment can be magnified by scaling the corresponding polynomial, proving the converse.

3.5 Fortet–Mourier refinements and sharp weighted order two

Proof of proposition 1.4 and (1.23). The convenient bounds follow from $\mathrm{osc}(h) \le 2B$, $\|h'\|_\infty \le L$. At a fixed value $h(x) = \alpha$, the sign of $K^{0,0}$ is positive left of x and negative right of x. The clipped affine function $u_{\alpha,x}$ is the largest permissible function on the left and the smallest on the right. It therefore maximises $f_h(x)$ under that fixed value. Optimise over $\alpha \in [-B,B]$ and use symmetry under $h \mapsto -h$ to obtain (1.20).

The sign and mass of $K^{1,0}$ show that q_x is a probability density and $f'_h(x) = (bx)^{-1}\int(h(x)-h(v))q_x(v)\,dv$. The largest permissible difference is $\min(L|v-x|, 2B)$ and is attained simultaneously for all v by $h(v) = B - \min(L|v-x|, 2B)$. This proves (1.22).

For sharpness of $\|xf''_h\|_\infty \le 2L/b$, first fix a small $x > 0$. Choose h' equal to $-L$ on $(0, x-\varepsilon)$, equal to $+L$ near x, and zero beyond $x+2\varepsilon$, with transitions on intervals of length at most a constant times ε. Its primitive can be shifted so that its norm is at most $L(x+4\varepsilon)/2$.

For small enough x, ε, it therefore belongs to $\mathcal{H}_{B,L}$. Write $w_x(v) = -bxK^{2,1}(x,v)p(v)$, a probability density. Its mass on the right of x is

$$bxS''(x)\overline{P}_2(x) \longrightarrow 0 \quad (x \downarrow 0),$$

because $S''(0+) < \infty$ and $\overline{P}_2(0+) = ab$. For each fixed x, let $\varepsilon \downarrow 0$ first; the transition regions then have vanishing w_x-mass. The left mass tends to one as $x \downarrow 0$, and (2.17) gives

$$bxf_h''(x) = h'(x) - \int h'(v)w_x(v)\,dv \longrightarrow 2L.$$

This proves weighted sharpness for every $B > 0$ and the infinite unweighted class supremum.

For the sum-norm convention, maximise $\min(\alpha B, \beta L)$ over $B + L \le 1$. The value is $\alpha\beta/(\alpha+\beta)$, proving (1.24) and the weighted first-derivative factor. The preceding order-two tests have norm tending to zero; rescale by $\|h\|_\infty + \mathrm{Lip}(h)$ to obtain the sharp constant $2/b$ for the sum-norm unit class as well. □

4 Why the missing bounds cannot hold

4.1 Polynomial freedom at low derivative orders

Fix $r \ge 2$ and $0 \le n \le r-2$. Let $g(x) = Mx^n$ and define $h(x) = bxg'(x) + (ab - x)g(x)$. Gamma integration by parts gives $\mathbb{E}h(G) = 0$, and the endpoint conditions show that $f_h = g$. But h has degree $n+1 < r$, so $h^{(r)} = 0$, whereas $f_h^{(n)} = Mn!$. Letting $|M| \to \infty$ rules out a finite pointwise envelope. Any positive weight at an interior point preserves this obstruction.

4.2 Local smooth counterexamples at high derivative orders

Fix $x_0 > 0$, $j \ge 0$, and $n > j+1$. Choose $\psi \in C_c^\infty(-1,1)$ with $\psi^{(n)}(0) \ne 0$. For $0 < \delta < x_0/2$, let

$$g_\delta(x) = \delta^{j+1}\psi((x-x_0)/\delta), \qquad h_\delta(x) = bxg_\delta'(x) + (ab-x)g_\delta(x). \tag{4.1}$$

Integration of $(bxpg_\delta)' = ph_\delta$ gives $\mathbb{E}h_\delta(G) = 0$ and $f_{h_\delta} = g_\delta$. On this fixed interior neighbourhood,

$$\sup_\delta \|h_\delta^{(j)}\|_\infty < \infty, \qquad f_{h_\delta}^{(n)}(x_0) = \delta^{j+1-n}\psi^{(n)}(0).$$

A fixed rescaling normalises the controlled norm and the derivative diverges. Use $j = 0$ for bounded tests, $j = 1$ for Wasserstein, and $j = r$ for $\mathcal{Z}_r$. In the case $j = 1$, $\|h_\delta\|_\infty = O(\delta)$, so the same construction fits either Fortet–Mourier convention after rescaling. It also rules out finite weighted factors for any continuous weight positive at x_0.

4.3 Boundary divergence is a different obstruction

The last construction makes a derivative arbitrarily large at a fixed interior point. In contrast, the envelopes $1/(bx)$ and $2/(bx)$ are finite at every such point; it is their limit at zero that prevents an ordinary factor. Weighting by x repairs exactly that divergence. For higher Kolmogorov derivatives, the two endpoint requirements conflict for every power weight, leading instead to (1.15).

5 Evaluating and using the bounds

5.1 Numerical shape constants and explicit fallback bounds

a	u_a	v_a	$c_{a,1}$	$d_{a,1}$	$c_{a,2}$	$d_{a,2}$
0.5	2.0000	1.0000	1.3333	2.0000	0.8000	2.0000
1	1.0000	1.0000	1.0000	2.0000	0.6667	2.0000
2	0.5418	1.1516	0.6667	2.0000	0.5000	2.0000
5	0.3014	1.5720	0.3805	2.3116	0.2975	2.0886
10	0.2053	2.1033	0.2604	2.8962	0.2041	2.4619
50	0.0892	4.4859	0.1135	5.8094	0.0891	4.6454

Table 3: Illustrative numerical estimates of the exact shape constants. The optimisation includes the endpoint limits. Values are rounded to four decimals and are not certified upper bounds. For rigorous constants use the defining suprema or the proved analytic upper bounds.

The first two diagonal ordinary factors are often much smaller than the shape-independent bounds 2 and 1. The weighted constants must be optimised separately: multiplying an unweighted factor by a typical value of x is not a valid substitute.

For completeness, entirely explicit finite upper bounds for u_a and v_a can be obtained without optimisation. Work at $b = 1$ and write $S_a = P_{a,1}/(yp_{a,1})$, $R_a = \overline{P}_{a,1}/(yp_{a,1})$. For any $t > (a-1)_+$,

$$u_a \le \max\{S_a(t), R_a(t)\}, \qquad v_a \le \max\left\{tS_a(t), \frac{1}{1-(a-1)_+/t}\right\}. \tag{5.1}$$

For example $t = 2\max(1, a)$ makes the second term in the bound for v_a at most 2. These bounds may be conservative for large shapes. To prove them, use $\mathcal{A} \le S_a$ below t, $\mathcal{A} \le R_a$ above t, and monotonicity of the Mills ratios. For $y \ge t$,

$$\frac{\overline{P}_{a,1}(y)}{p_{a,1}(y)} = \int_0^\infty (1+s/y)^{a-1} e^{-s}\, ds \le \frac{1}{1-(a-1)_+/y},$$

which bounds $y\mathcal{A}(y)$ as asserted.

5.2 Stable evaluation and scaling

The recurrences (2.2)–(2.3) are convenient at moderate arguments. Close to zero or deep in a tail, subtraction may destroy accuracy. The positive integrals (2.4)–(2.5) or direct truncated moments avoid that problem. One useful special-function form, at scale one, is

$$S_a^{(n)}(y) = \mathrm{B}(a, n+1)\, {}_1F_1(n+1; a+n+1; y), \tag{5.2}$$

$$(-1)^n R_a^{(n)}(y) = n!\, U(n+1, a+n+1, y), \tag{5.3}$$

$$\mathcal{D}_{a,1,n}(y) = \frac{2y^{n+1}p_{a,1}(y)}{n!} S_a^{(n)}(y)(-1)^n R_a^{(n)}(y). \tag{5.4}$$

Here B is the beta function, ${}_1F_1$ the confluent hypergeometric function, and U Tricomi's function. Their standard integral definitions give these formulas directly. Logarithmic or scaled evaluation is preferable when individual factors overflow.

Scale conversion should be made at the operator level. Put $y = x/b$ and $g(y) = h(by)$. Then

$$f_{h;a,b}(x) = b^{-1} f_{g;a,1}(y), \qquad f^{(n)}_{h;a,b}(x) = b^{-n-1} f^{(n)}_{g;a,1}(y). \tag{5.5}$$

Since $\|g^{(r)}\|_\infty = b^r \|h^{(r)}\|_\infty$, an (n, r) bound with weight x^q acquires the factor $b^{r+q-n-1}$. Oscillation is unchanged. For Fortet–Mourier tests the parameters become (B, bL) before applying (5.5). If the operator is instead $xf'(x) + (a - \lambda x)f(x)$ with rate $\lambda = 1/b$, its solution is $bf_{h;a,b}$, so all our solution bounds must be multiplied by b.

5.3 How the weights enter approximation arguments

For positive X and integrable terms, the Stein identity is

$$\mathbb{E}h(X) - \mathbb{E}h(G) = \mathbb{E}\{bXf'_h(X) + (ab - X)f_h(X)\}. \tag{5.6}$$

For discontinuous tests evaluated at atoms, use the Stein-equation version $f'_{h,\mathrm{S}}(x) = \{h(x) - \mathbb{E}h(G) + (x - ab)f_h(x)\}/(bx)$. It agrees with the ordinary derivative almost everywhere and obeys the bounded-test envelope everywhere. At a Kolmogorov threshold it is the left derivative. This convention does not create classical higher derivatives at a jump.

If a coupling or integration-by-parts calculation leaves a term $\mathbb{E}[TXf''_h(X)]$, the Wasserstein bound is

$$|\mathbb{E}[TXf''_h(X)]| \le \frac{2L}{b}\mathbb{E}|T|.$$

Replacing this with an unweighted bound on f''_h would fail. More generally, a term $\mathbb{E}[TXf^{(r+1)}_h(X)]$ under $H_r < \infty$ has the same factor $2H_r/b$. At derivative order r, retaining the envelope yields

$$|\mathbb{E}[TXf^{(r)}_h(X)]| \le H_r\mathbb{E}[|T|X\mathcal{D}_{a,b,r}(X)] \le d_{a,r}H_r\mathbb{E}|T|.$$

This is the role of the weighted bounds in moment-matched Gamma approximation. In a Kolmogorov argument, a Taylor expansion crossing the threshold must separately retain the jump term.

References

[1] G. Bailly, F. Rapin, Y. Swan and R. von Sachs (2026). *Explicit Gamma–Stein Bounds for Satterthwaite's Approximation*. Preprint, version 1, 15 September 2026. arXiv:2609.17880.